Image and Vision Computing

Projection

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Projective Geometry

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

- NOTE: If the scalar is 1, there is no need for the multiplication!

Example:

$$(2,3) \rightarrow (2,3,1) \sim (4,6,2) \sim (-4,-6,-2)\ldots$$

$$(3,-1,2) \rightarrow (3,-1,2,1) \sim (6,-2,4,2) \sim (-6,2,-4,-2)\ldots$$

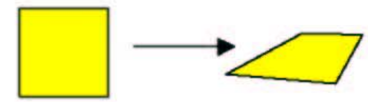
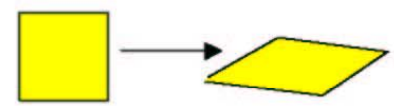
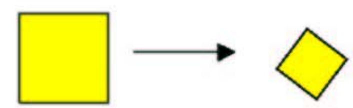
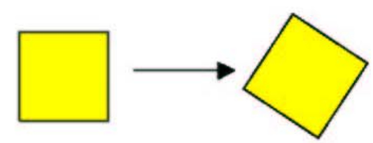
Back to Cartesian Coordinates:

- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

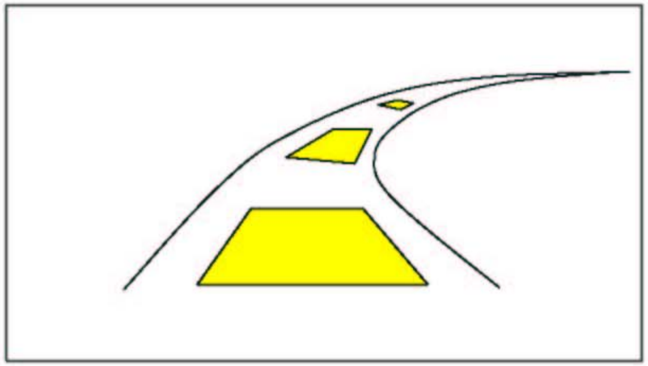
$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

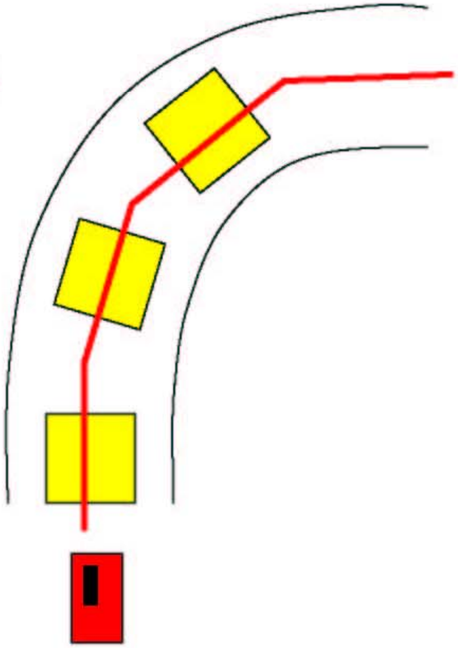
Special Projectivities

		Invariants	
Projectivity 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	Collinearity, Cross-ratios	
Affine transform 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_x \\ 0 & 0 & 1 \end{bmatrix}$	Parallelism, Ratios of areas, Length ratios	
Similarity 4 dof	$\begin{bmatrix} s\,r_{11} & s\,r_{12} & t_x \\ s\,r_{21} & s\,r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	Angles, Length ratios	
Euclidean transform 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$	Angles, Lengths, Areas	

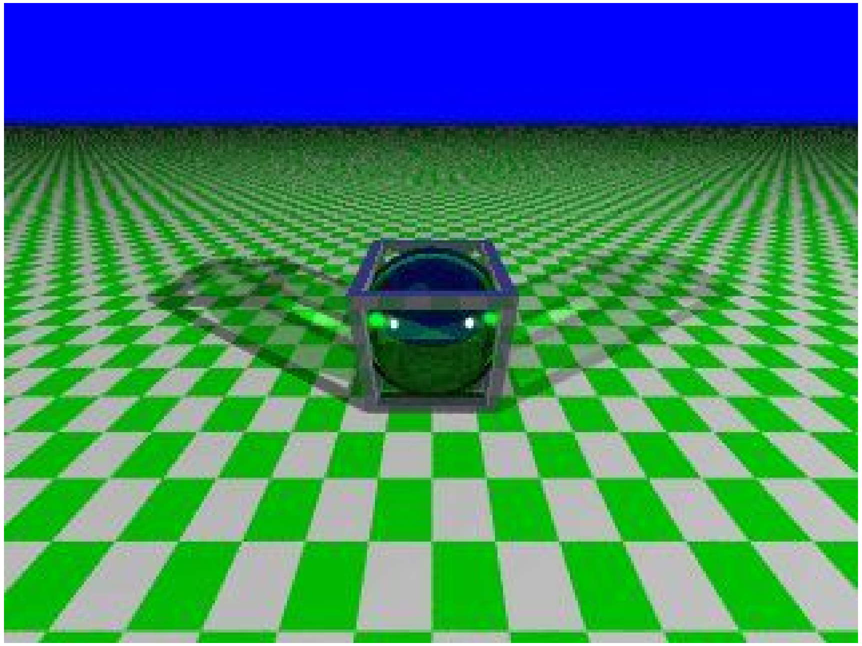
Example of Application

- Robot going down the road
- Large squares painted on the road to make it easier
- Find road shape without perspective distortion from image
  - Use corners of squares: coordinates of 4 points allow us to compute matrix **H**
  - Then use matrix **H** to compute 3D road shape





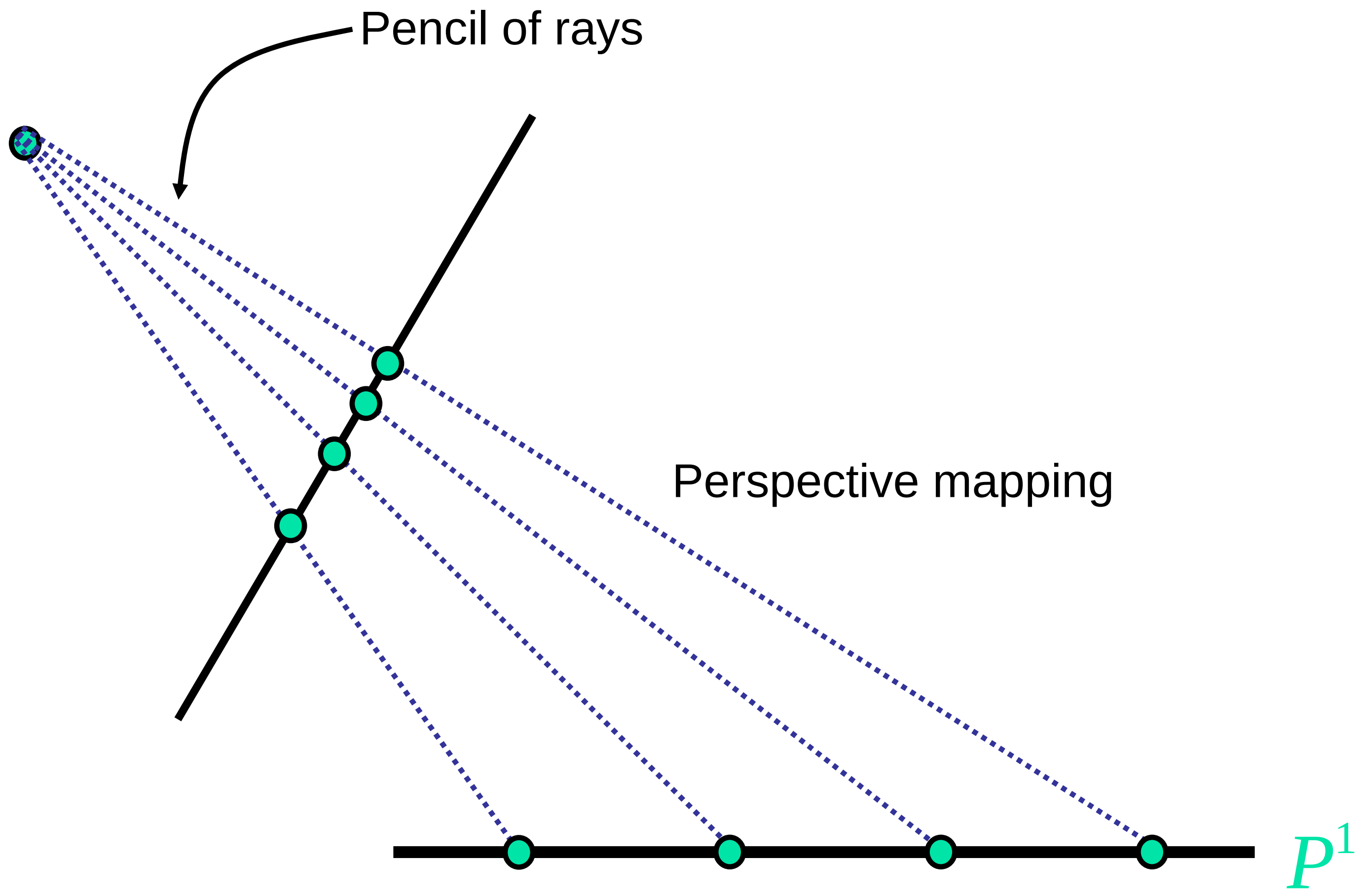
Perspective Distortion



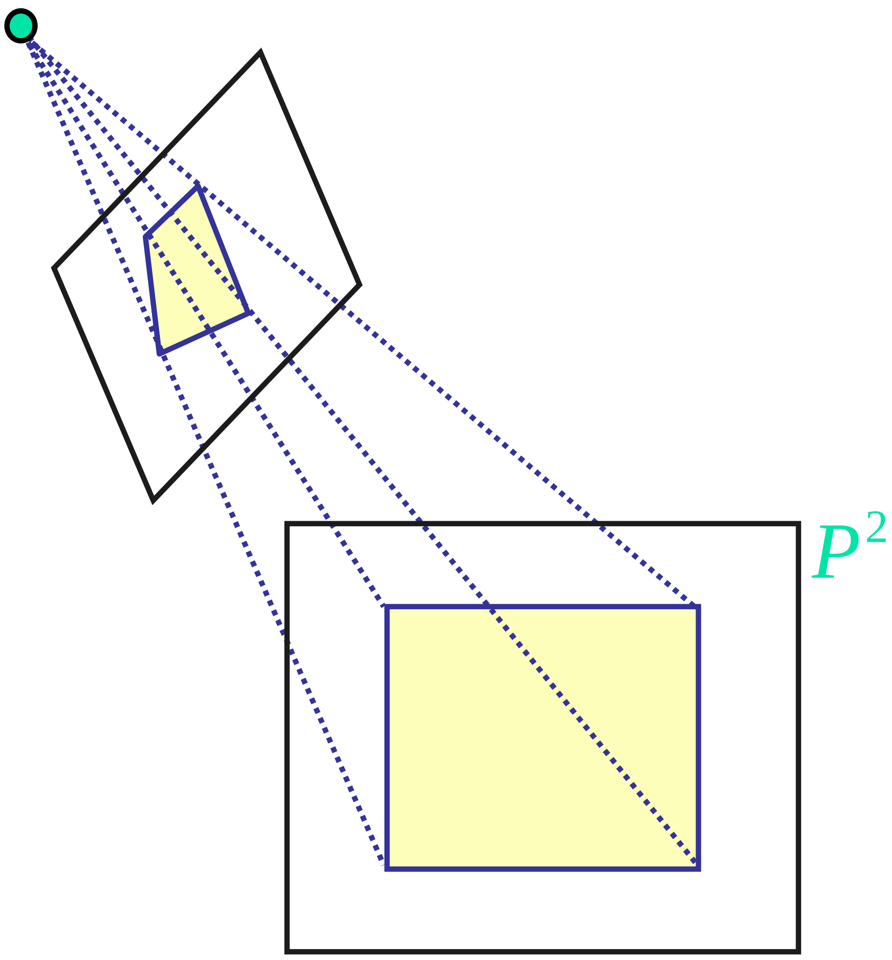
Q: Where do parallel lines meet?

A: Parallel lines meet at the horizon (“vanishing line”)

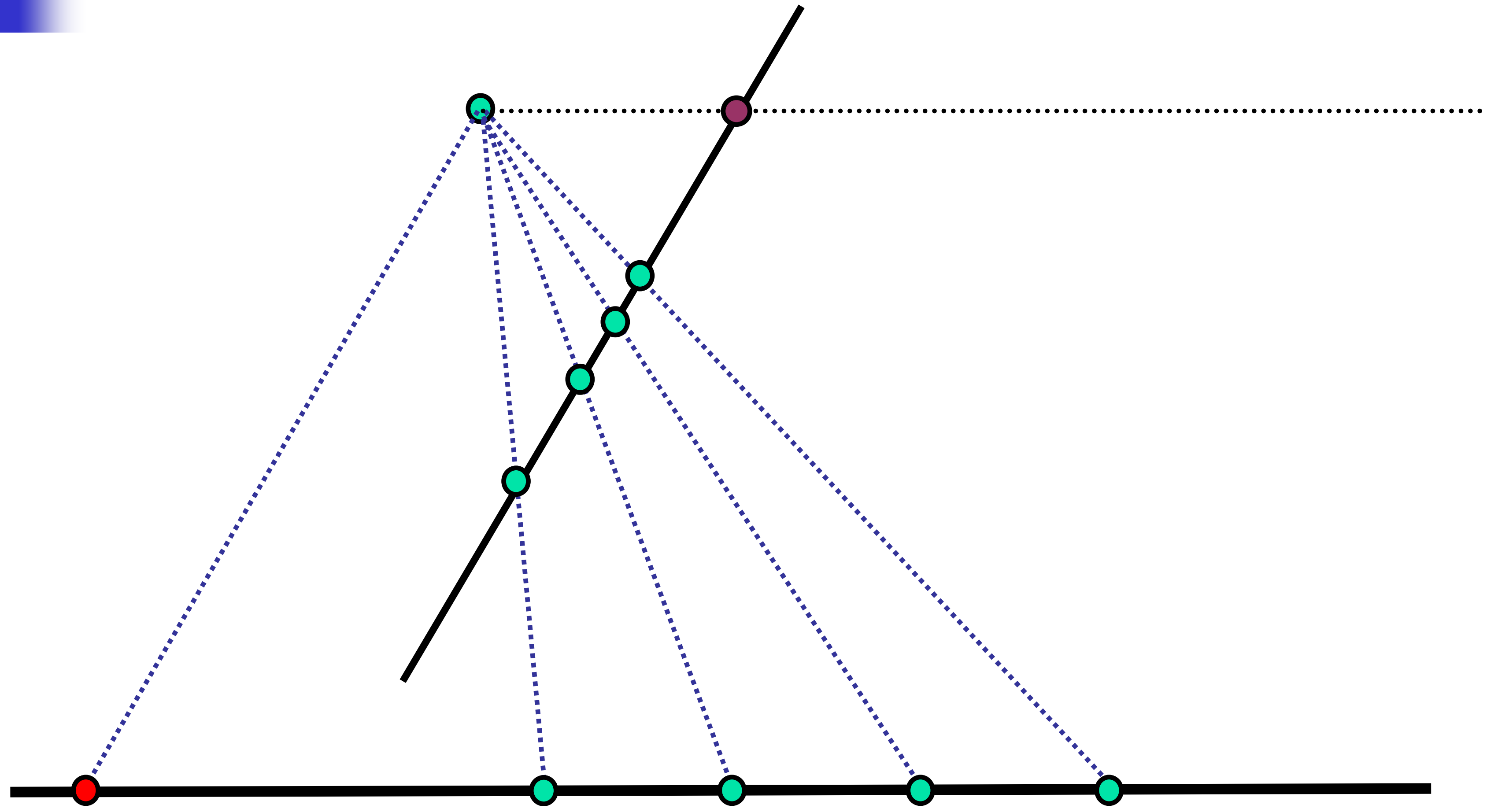
Line Perspective



Plane Perspective

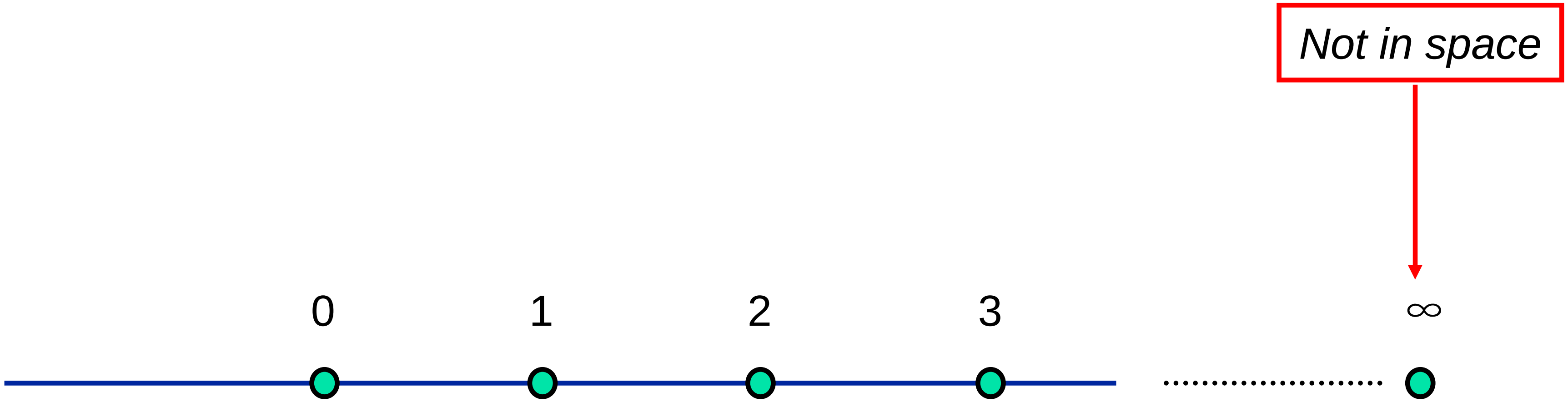


Ideal points



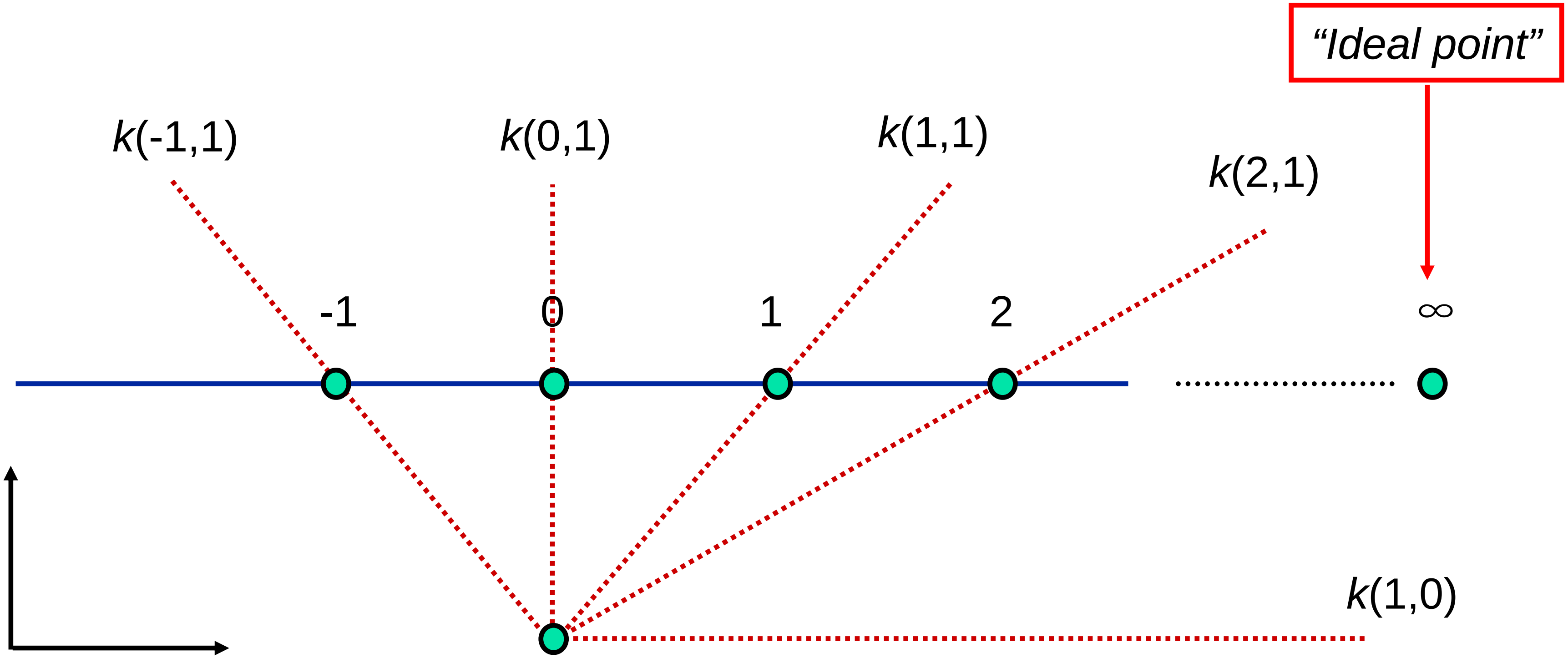
Projective transformation can map ∞ to a real point

Coordinates in Euclidean Space

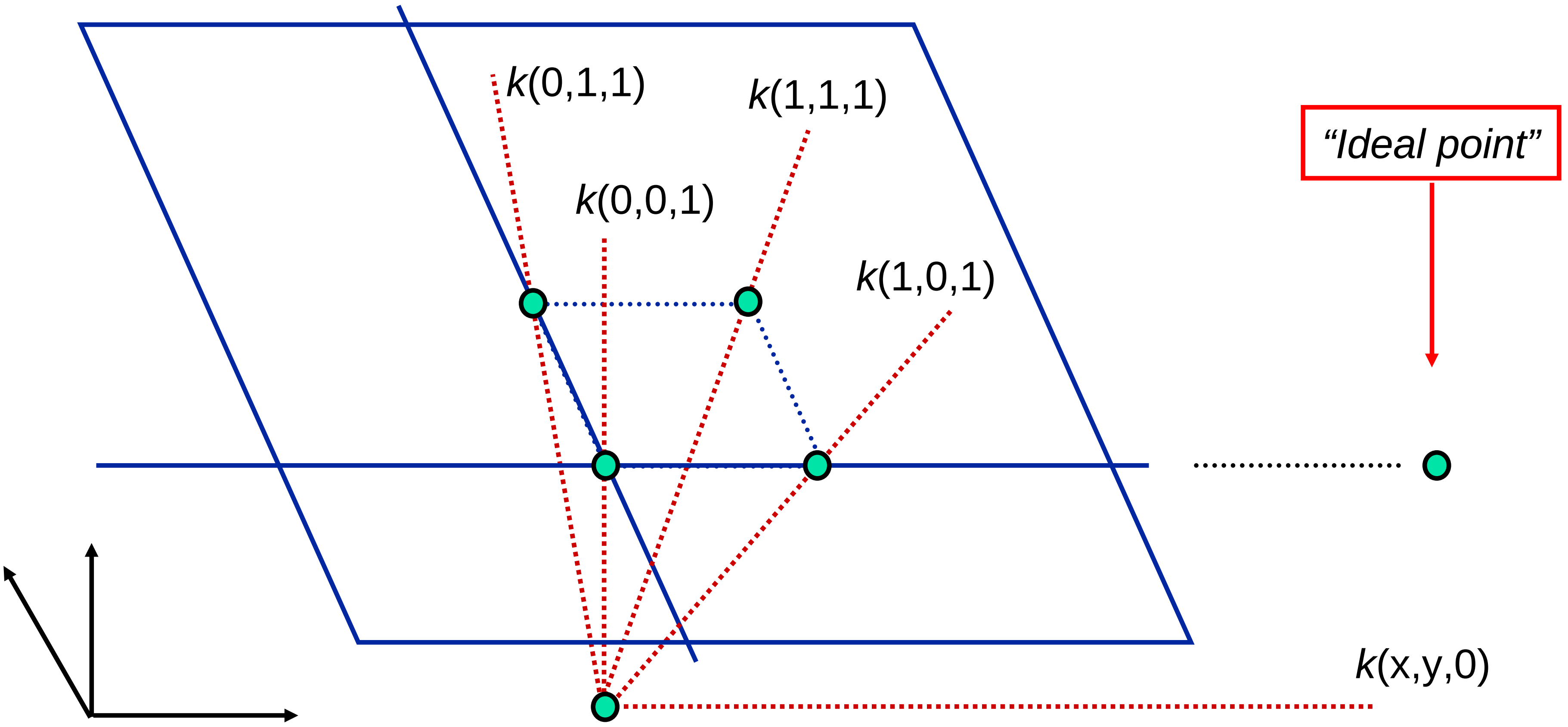


Coordinates in Projective Line

Points on a line P^1 are represented as rays from origin in 2D,
Origin is excluded from space

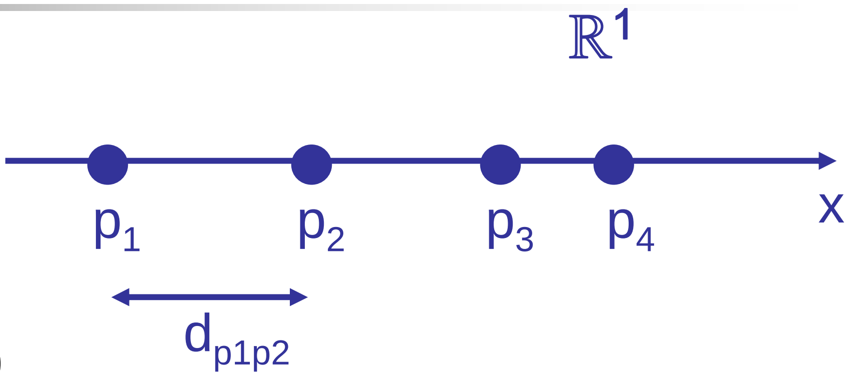


Coordinates in Projective Plane



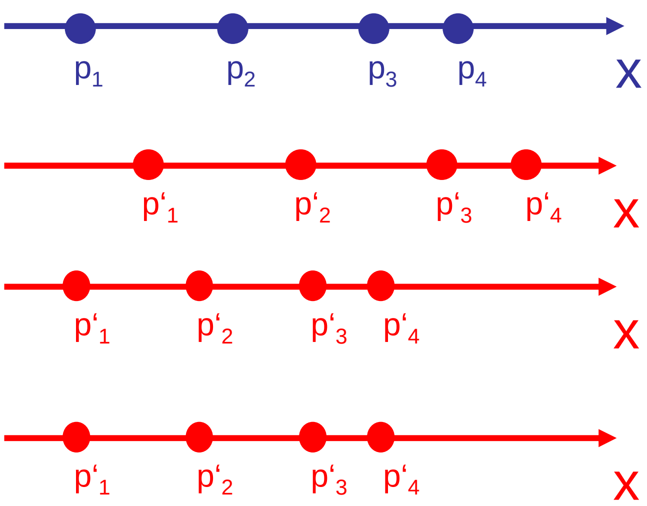
1D Projective Geometry

- Euclidean 1D vector space:



  - Geometric Primitives: points
  - Representation: 1D vector: p = (x)

  - Transformations:
    - Translation: p' = (x+t)
    - Scaling: p' = (s*x)
    - Translation and Scaling: p' = (s*x+t)

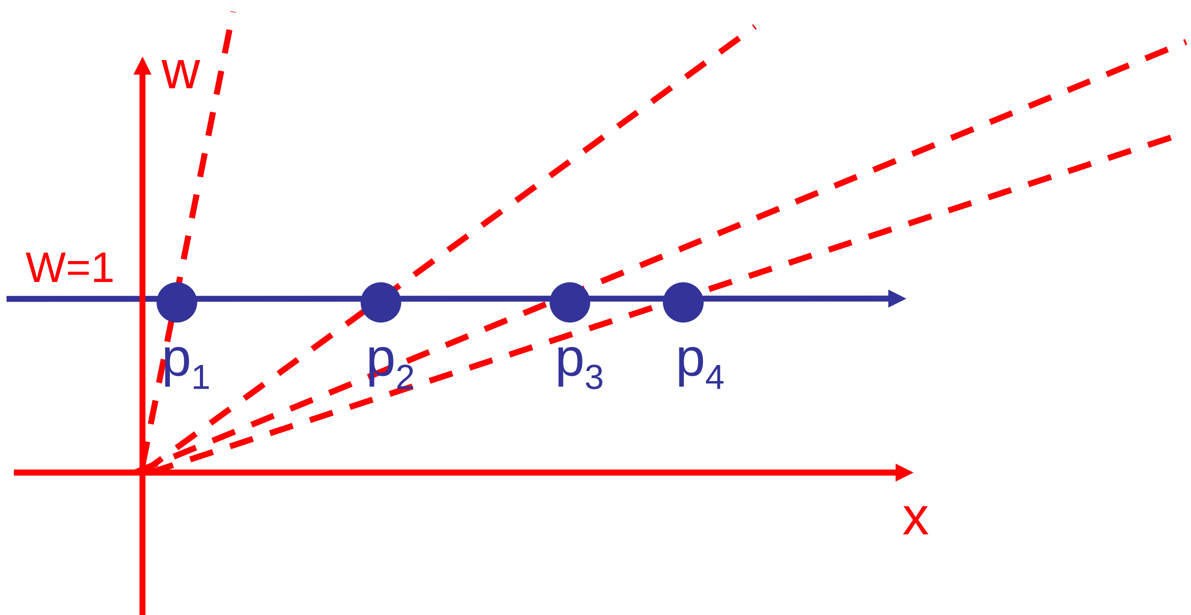


1D Projective Geometry $\mathbb{P}^1$

- 1D Projective space:

Representation: 1D vector: $p' = (x' \ w')^T$, where $p = (x) = (x'/w')$

e.g. $p' = (2\ 1)^T = (4\ 2)^T = (10,5)^T$ are all equivalent to $p = (2)$



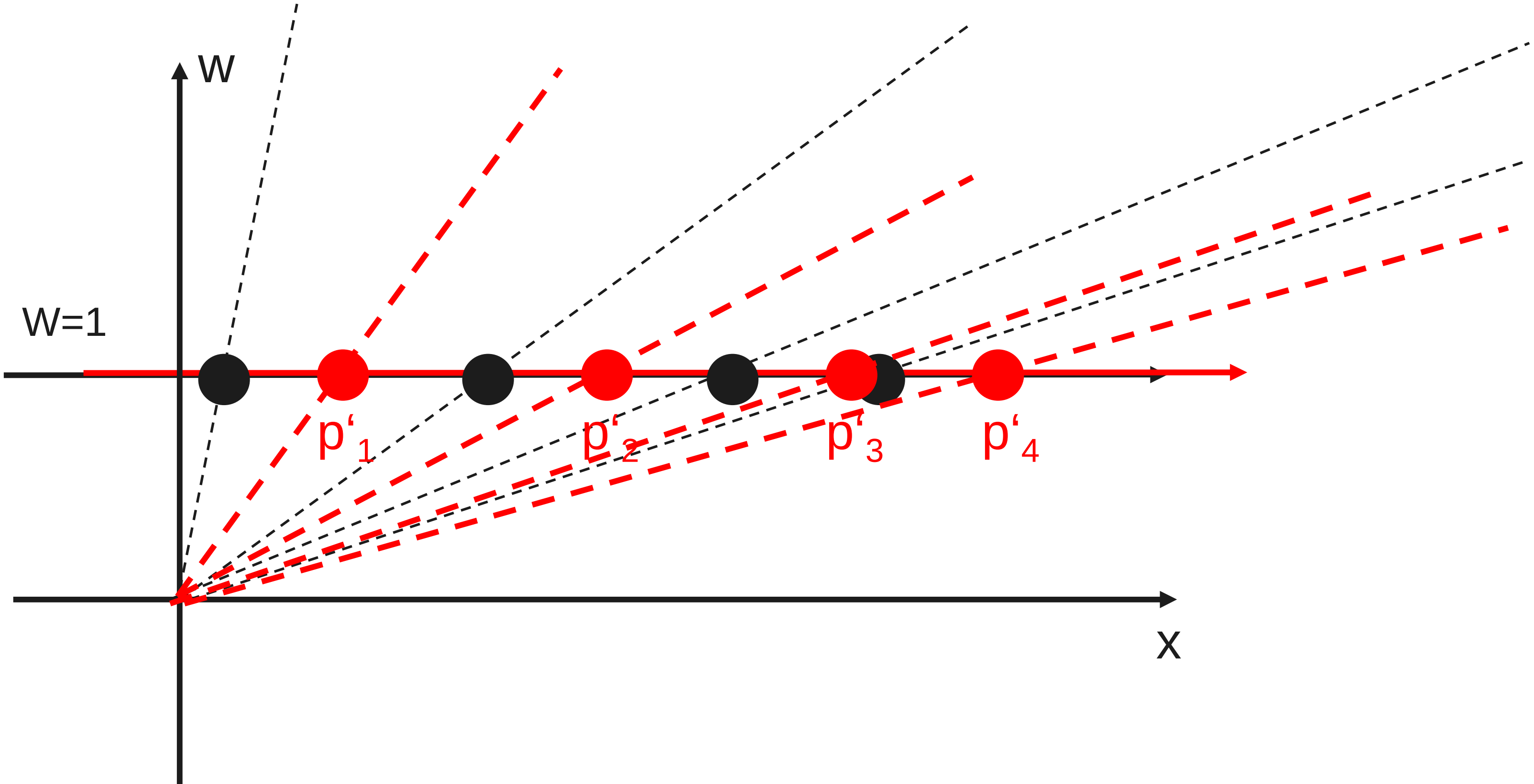
1D Projective Geometry

- 1D Projective space:
  - Geometric Primitives: points
  - Representation: 1D vector: p = $(x\ w)^T$
  - Transformations:
    - Translation: p‘ = T * p, 1DOF $T = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
    - Scaling: p‘ = S * p, 1DOF $S = \begin{bmatrix} 1 & 0 \\ 0 & 1/s \end{bmatrix}$
    - Translation and Scaling: p‘ = M * p, 2DOF $M = \begin{bmatrix} 1 & t \\ 0 & 1/s \end{bmatrix}$
    - Projective Mapping: p‘ = P * p, 3DOF

1D Projective Geometry

- Translation:

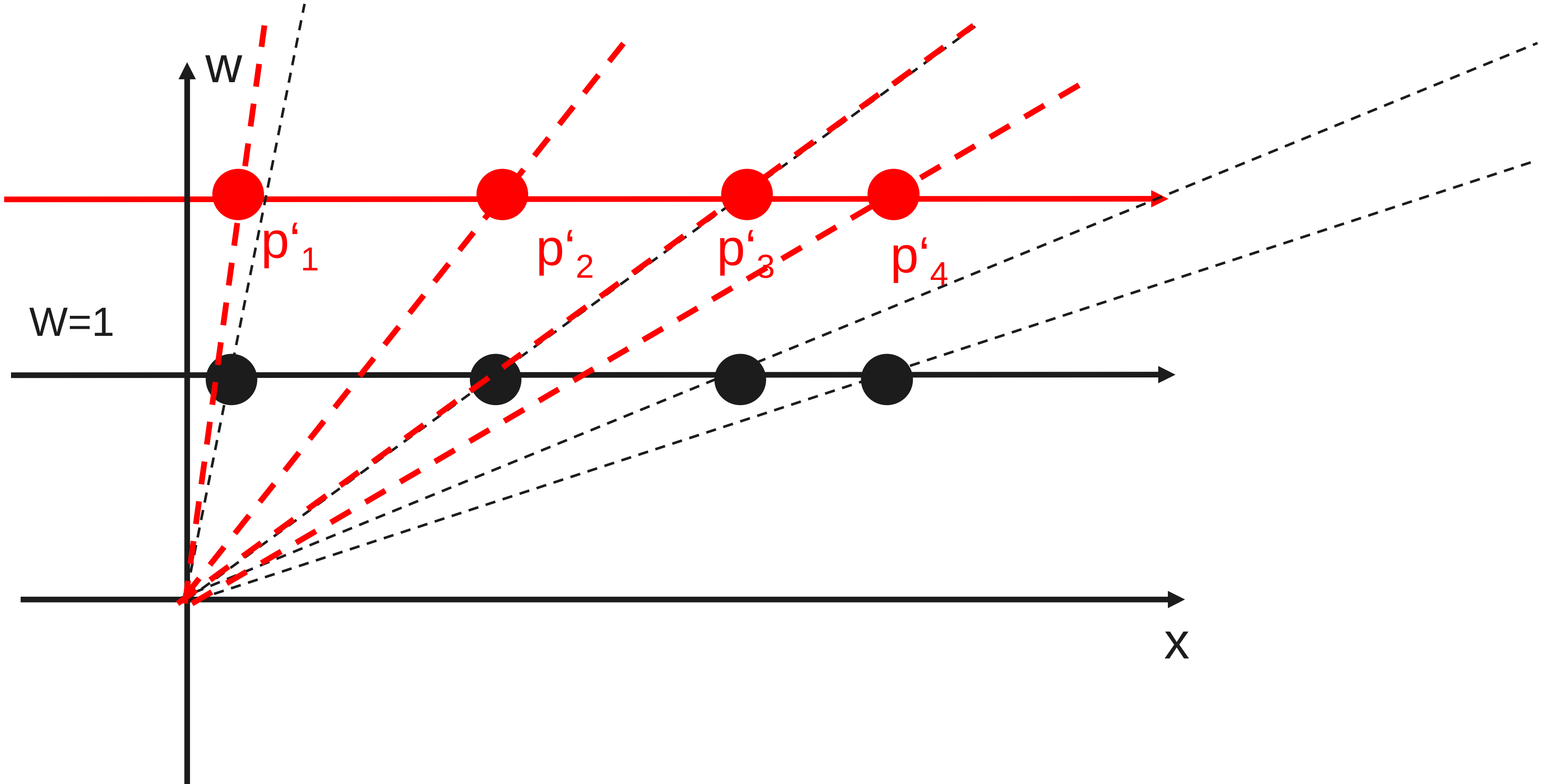
$$p' = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} p$$



1D Projective Geometry

- Scaling:

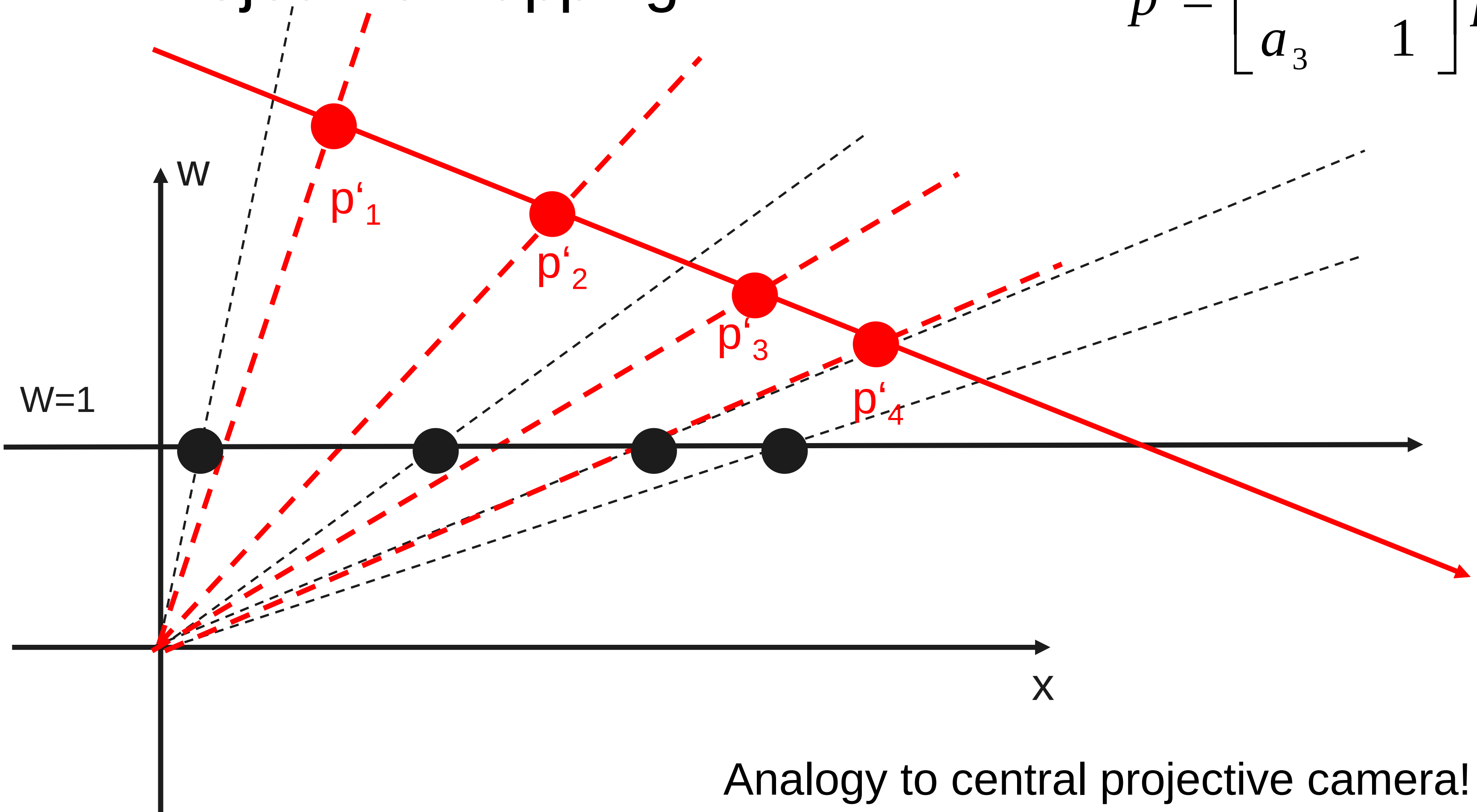
$$p' = \begin{bmatrix} 1 & 0 \\ 0 & 1/s \end{bmatrix} p$$



1D Projective Geometry

- Projective Mapping:

$$p' = \begin{bmatrix} a_1 & a_2 \\ a_3 & 1 \end{bmatrix} p$$



Analogy to central projective camera!

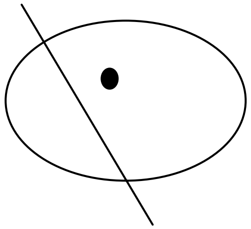
1D Projective Geometry

- Invariants:
  - transformations form a hierarchy, some geometric features remain unchanged under the transformations
  - Translation (isometry)
    - Length, overall scale
  - Translation and Scaling (similarity)
    - Ratio of lengths (d12 : d23)
  - Projective Mapping (projectivity)
    - Cross ratio (ratio of ratios):

      d12*d34/d13*d24

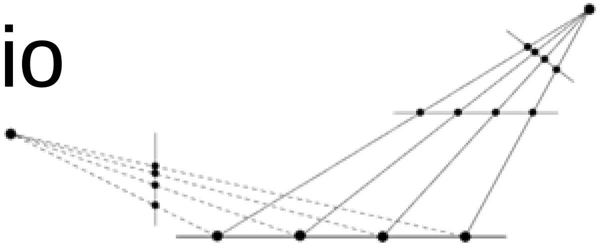
Projective 2D Geometry

- Points, lines & conics

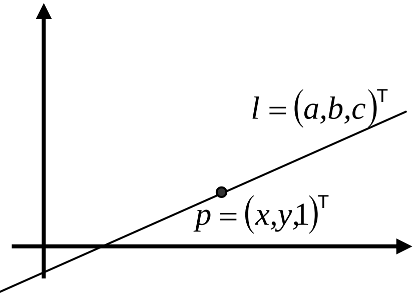


- Transformations & invariants

- 1D projective geometry and the Cross-ratio



Homogeneous coordinates



Homogeneous representation of lines

$$ax+by+c=0 \qquad (a,b,c)^{\mathsf{T}}$$

$$(ka)x+(kb)y+kc=0, \forall k\neq 0 \qquad (a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3-(0,0,0)^{\mathsf{T}}$ forms $\mathbf{P}^2$

Homogeneous representation of points

$\mathrm{x}=(x,y)^{\mathsf{T}}$ on $\mathrm{l}=(a,b,c)^{\mathsf{T}}$ if and only if $ax+by+c=0$

$$(x,y,1)(a,b,c)^{\mathsf{T}}=(x,y,1)\mathrm{l}=0 \qquad (x,y,1)^{\mathsf{T}} \sim k(x,y,1)^{\mathsf{T}}, \forall k\neq 0$$

The point x lies on the line l if and only if $\mathrm{x}^{\mathsf{T}}\mathrm{l}=\mathrm{l}^{\mathsf{T}}\mathrm{x}=0$

Homogeneous coordinates $(x_1,x_2,x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x,y)^{\mathsf{T}}$

Points and lines

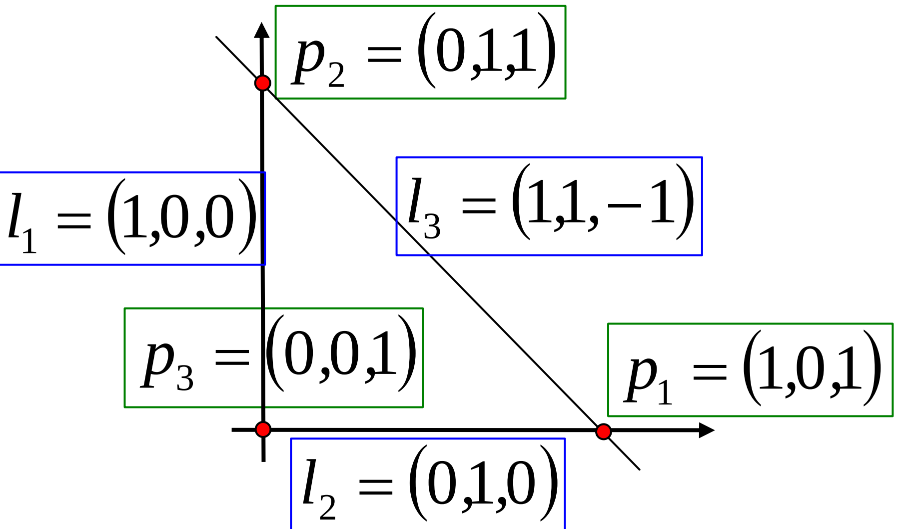
The point $p(x,y,1)^T$ lies on the line $l(a,b,c)^T$ if and only if $p^T l = l^T p = 0$

i.e. $ax+by+c=0$

The line l pass through two points $p_1(x_1,y_1,1)$ and $p_2(x_2,y_2,1)$ is homogeneously defined by l=p1xp2

Note that $(p_1 x p_2)^T p_1 = 0$, $(p_1 x p_2)^T p_2 = 0$

The intersection point p of two lines $l_1(a_1,b_1,c_1)$ and $l_2(a_2,b_2,c_2)$ is homogeneously defined by $p = l_1 x l_2$



We verify:

$$p_1 \cdot l_3 = 1\cdot 1 + 0\cdot 1 + 1\cdot -1 = 0$$

$$p_1 \times p_2 = (0\cdot 1 - 1\cdot 1, 1\cdot 0 - 1\cdot 1, 1\cdot 1 - 0\cdot 0)$$

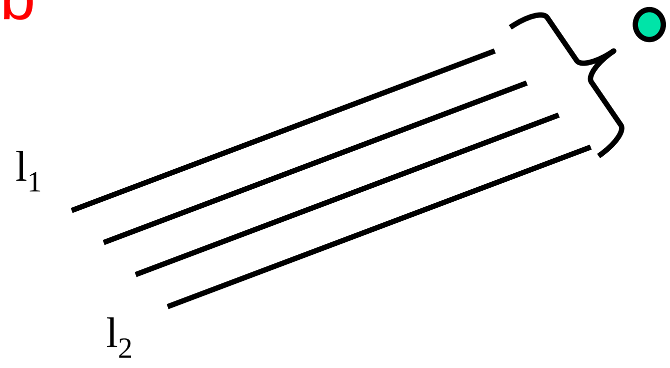
$$= (-1,-1,1) \propto (1,1,-1) = l_3$$

$$l_1 \times l_2 = (0\cdot 0 - 0\cdot 1, 0\cdot 0 - 1\cdot 0, 1\cdot 1 - 0\cdot 0)$$

$$= (0,0,1) = p_3$$

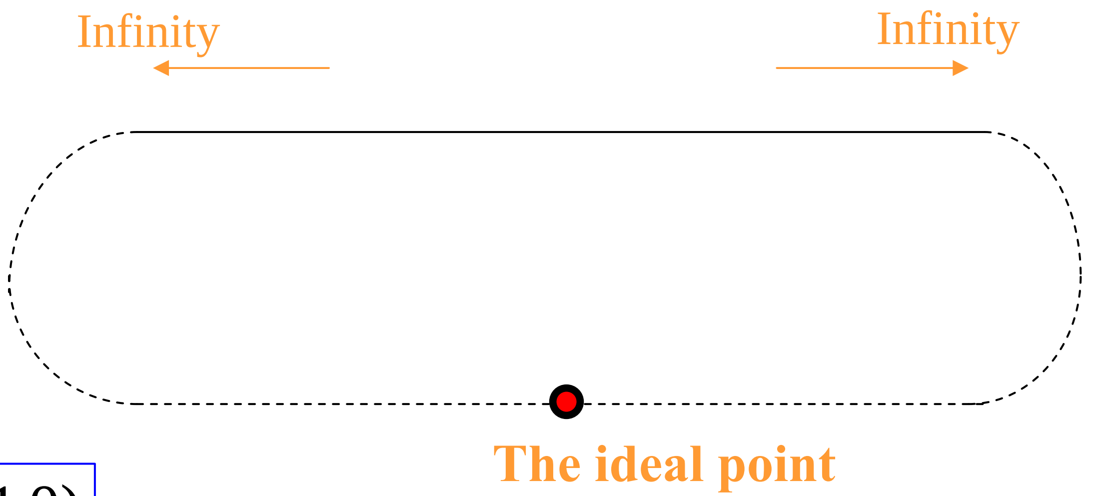
Ideal points

It is independent of the third coordinate c
It is solely dependent on the ratio a/b

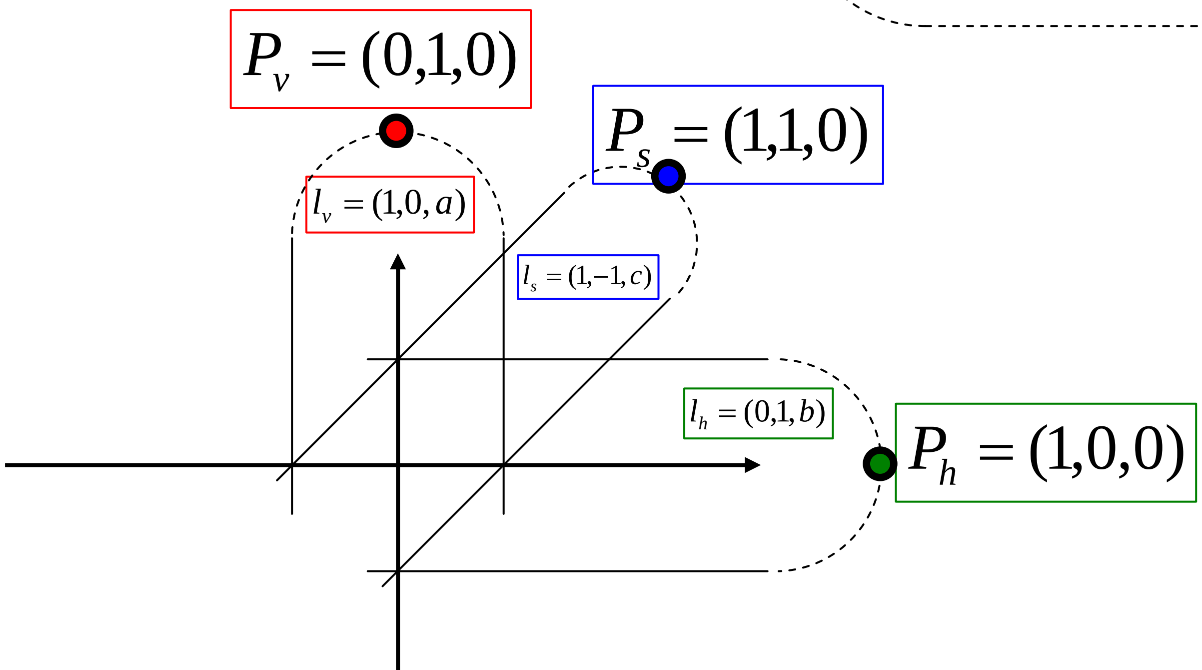


Q: How many ideal points are there in P^2?
A: 1 degree of freedom family – the line at infinity

Ideal Points



All parallel lines uniquely determine one ideal point



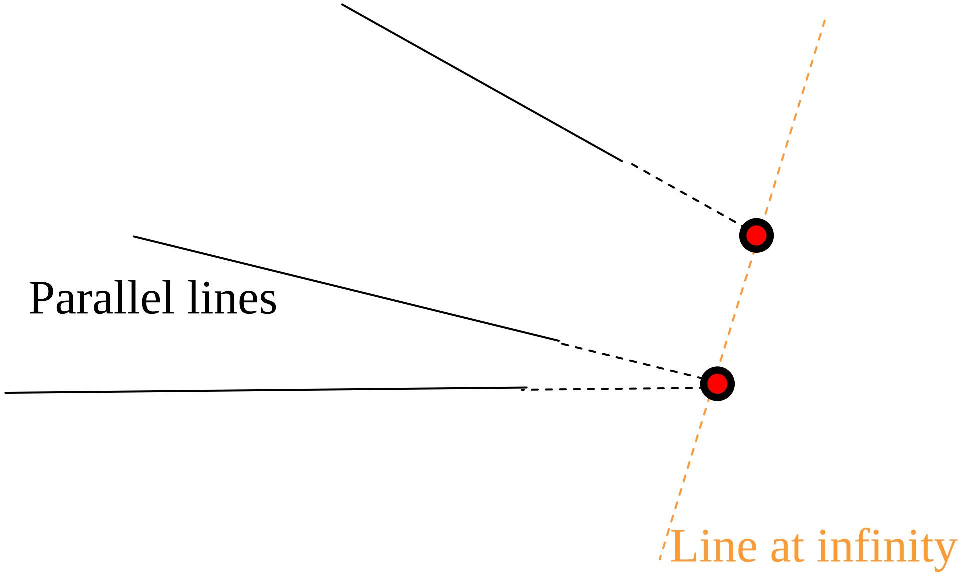
We validate:

$$l_v^T P_v = 0$$

$$l_h^T P_h = 0$$

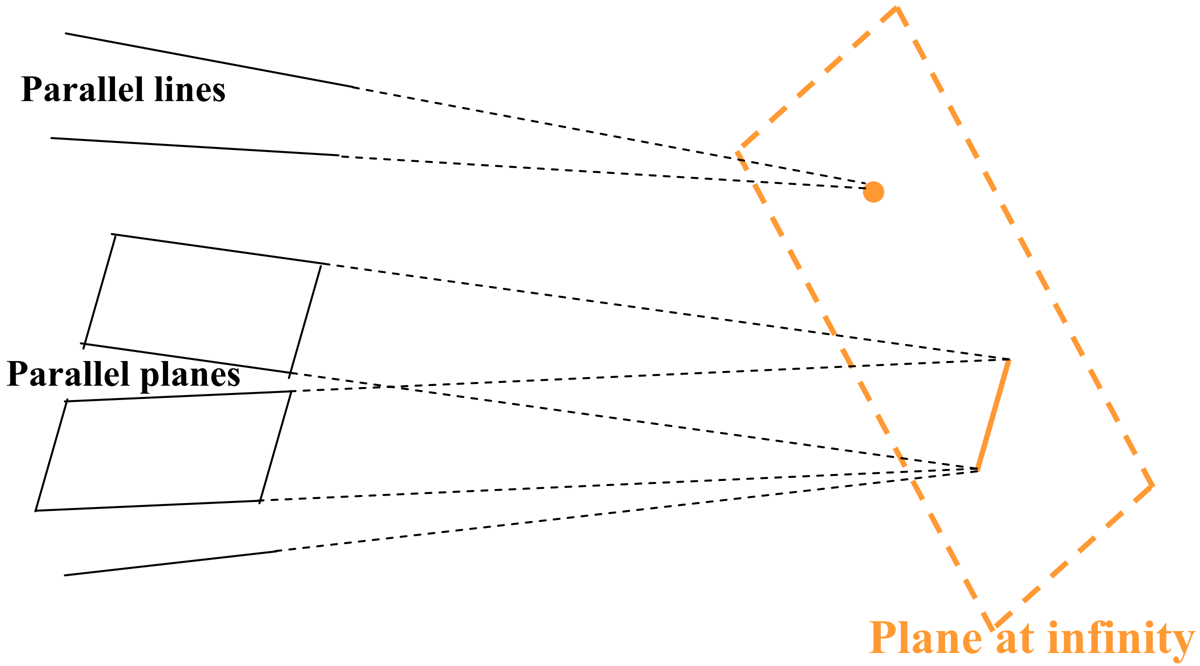
$$l_s^T P_s = 0$$

Line at infinity



- All ideal points of a 2D plane form an ideal line, which is called the line at infinity of this 2D plane.

Plane at infinity



- All ideal points of a 3D space form an ideal plane, which is called the plane at infinity of this 3D space.

Points from lines and vice-versa

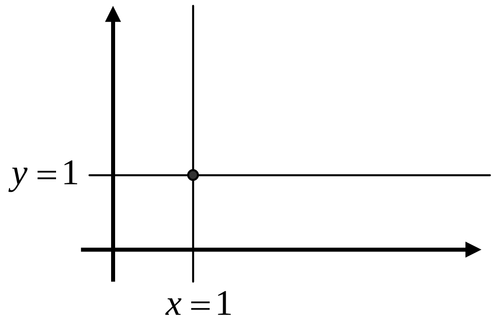
Intersections of lines

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Line joining two points

The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example



Ideal points and the line at infinity

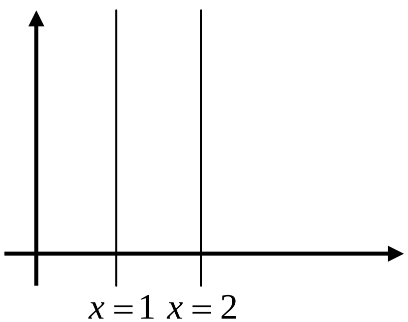
Intersections of parallel lines

$\mathrm{l} = (a, b, c)^\mathsf{T}$ and $\mathrm{l}' = (a, b, c')^\mathsf{T}$ $\quad\quad \mathrm{l} \times \mathrm{l}' = (b, -a, 0)^\mathsf{T}$

Example

$(b, -a)$ tangent vector

(a, b) normal direction



Ideal points $\quad (x_1, x_2, 0)^\mathsf{T}$

Line at infinity $\quad \mathrm{l}_\infty = (0,0,1)^\mathsf{T}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathrm{l}_\infty$$

Note that in **P**^{2} there is no distinction between ideal points and others

Practice

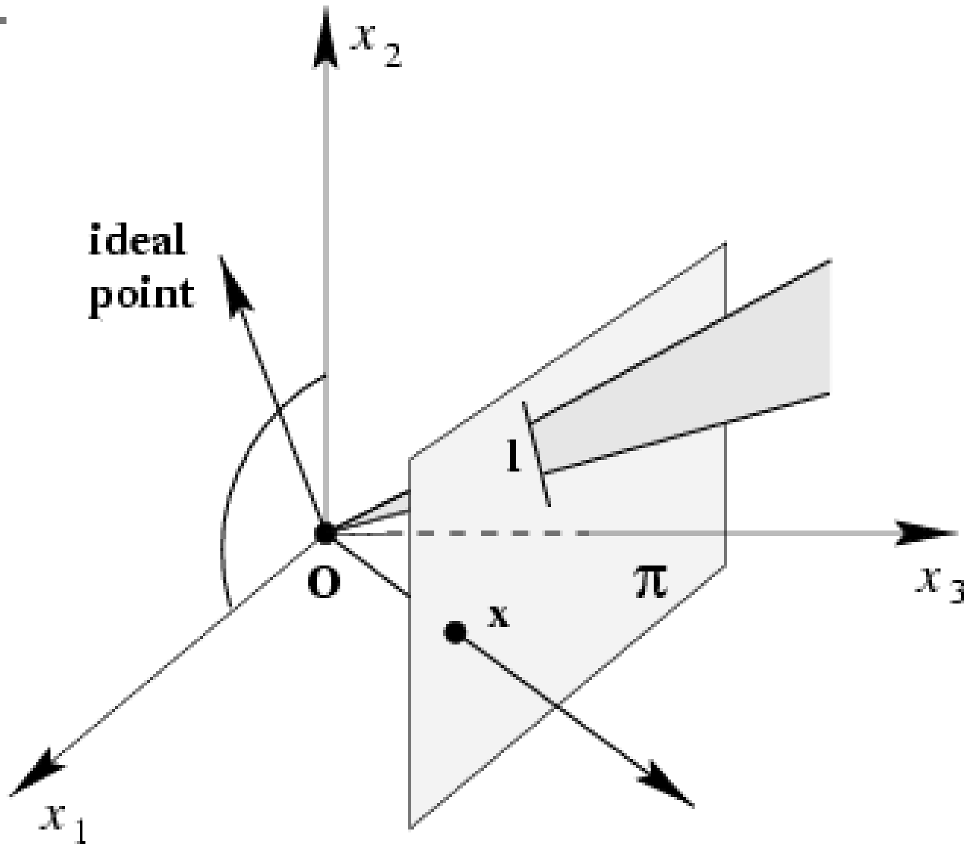
- All ideal points are on l_∞:
  - Proof: $(0,0,1)\bullet(x_1,x_2,0)^T = 0$
- Any line l intersects with l_∞ line at an ideal point
  - Proof: $(a,b,c)x(0,0,1) = (b,-a,0)$
- Two parallel lines l and l' always meet at an ideal point
  - Proof: Let $l = (a,b,c)^T$ and $l' = (a,b,c')^T$
  - ……

2D Projective Geometry: Basics

- A point: $(x, y, z)^T \in P^2 \quad \Leftrightarrow \quad (\frac{x}{z}, \frac{y}{z})^T \in \mathbb{R}^2$
- A line: $ax + by + cz = 0 \quad \Leftrightarrow \quad a(\frac{x}{z}) + b(\frac{y}{z}) + c = 0$

  we denote a line with a 3-vector $(a, b, c)^T$
- Points and lines are dual: *p* is on *l* if $l^T p = 0$
- Intersection of two lines: $l_1 \times l_2$,
- A line through two points: $p_1 \times p_2$

A model for the projective plane



exactly one line through two points

exaclty one point at intersection of two lines

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a:b:c:d:e:f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

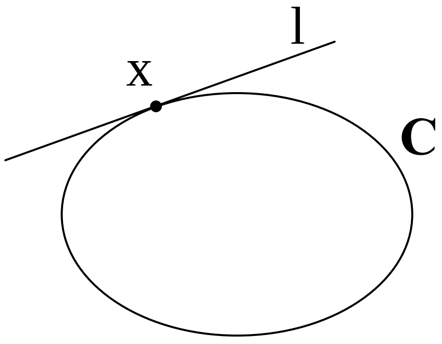
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Tangent lines to conics

The line l tangent to $\mathbf{C}$ at point x on $\mathbf{C}$ is given by $\mathrm{l}=\mathbf{C}\mathrm{x}$

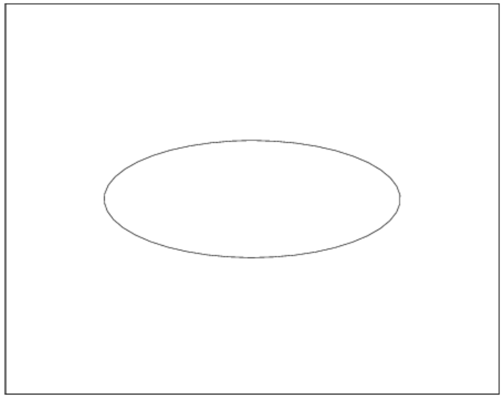


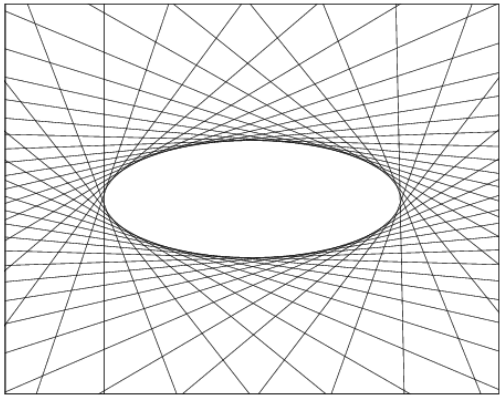
Dual conics

A line tangent to the conic $\mathbf{C}$ satisfies $\mathbf{l}^{\mathsf{T}} \mathbf{C}^{*} \mathbf{l} = 0$

In general ($\mathbf{C}$ full rank): $\mathbf{C}^{*} = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes

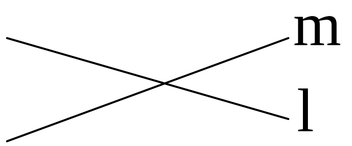




Degenerate conics

A conic is degenerate if matrix $\mathbf{C}$ is not of full rank

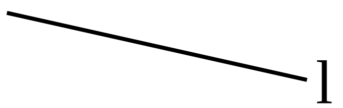
e.g. two lines (rank 2)



$$\mathbf{C} = \mathrm{l}\mathrm{m}^{\mathsf{T}} + \mathrm{m}\mathrm{l}^{\mathsf{T}}$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathrm{l}\mathrm{l}^{\mathsf{T}}$$



Degenerate line conics: 2 points (rank 2), double point (rank1)

Note that for degenerate conics $\left(\mathbf{C}^{*}\right)^{*} \neq \mathbf{C}$

Projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.

Theorem:

A mapping $h: P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix $\mathbf{H}$ such that for any point in P^2 reprented by a vector x it is true that $h(x) = \mathbf{H}x$

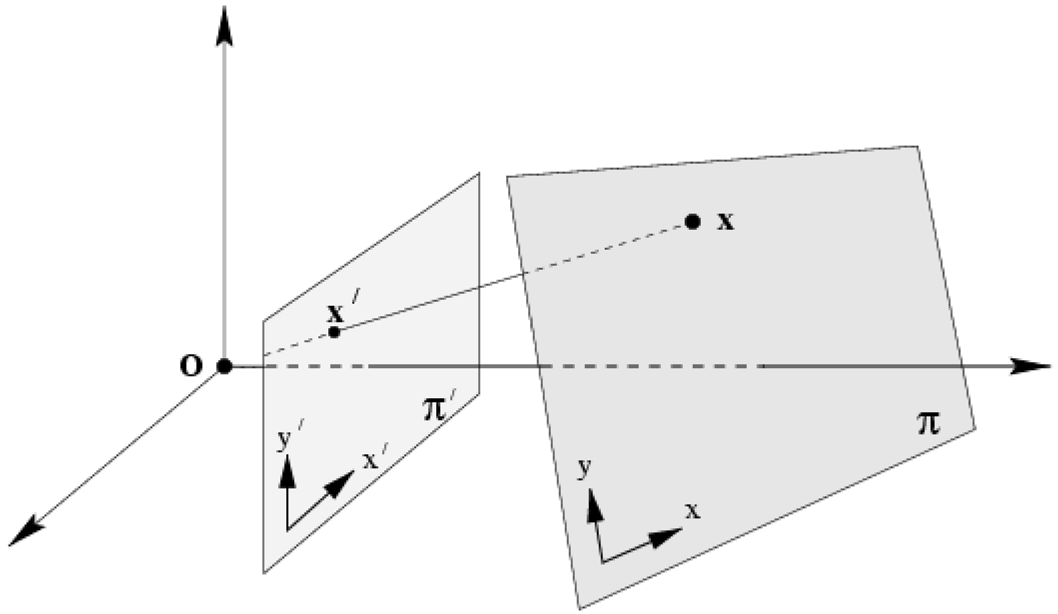
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad x' = \mathbf{H}\, x$$

8DOF

projectivity=collineation=projective transformation=homography

Mapping between planes



central projection may be expressed by x'=Hx

(application of theorem)

Removing projective distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

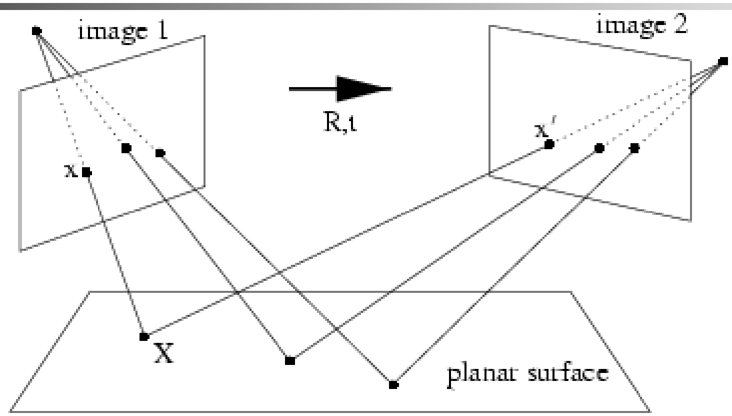
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

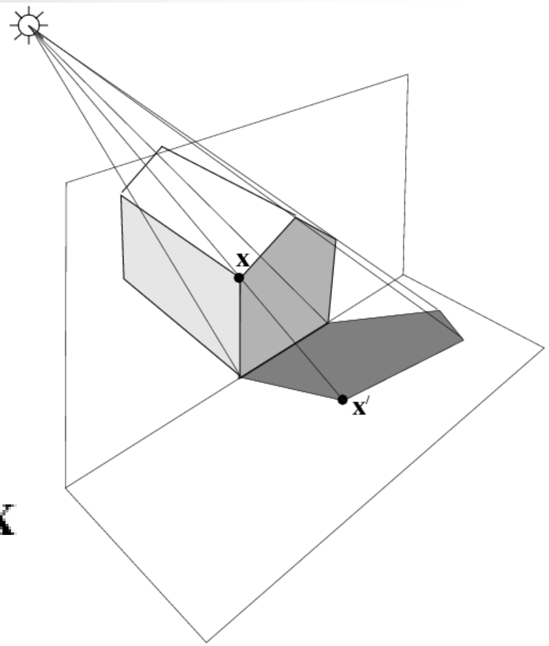
(linear in h_{ij})

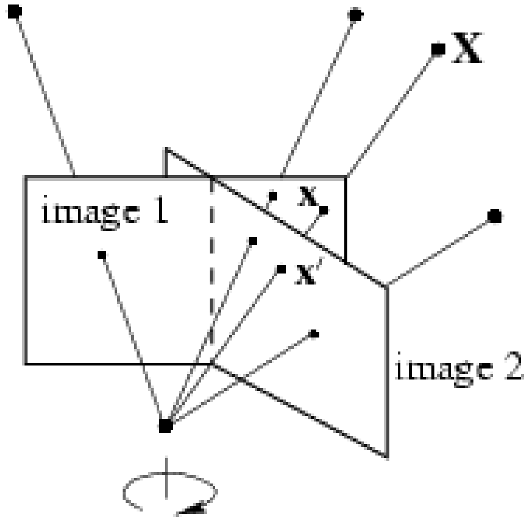
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary,
better ways to compute (see later)

More examples







Transformation of lines and conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

A hierarchy of transformations

Euclidean group (upper left 2x2 orthogonal)

Similarity groun (scaled Euclidean)

Affine group (last row (0,0,1))

Projective linear group (general)

- *Can be described algebraically*
  - *characterized by invertible 3x3 matrices*
  - *or in terms of invariants*

Class I: Isometries (*iso*=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$

orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities *(isometry + scale)*

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix}\mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also know as *equi-form* (shape preserving)

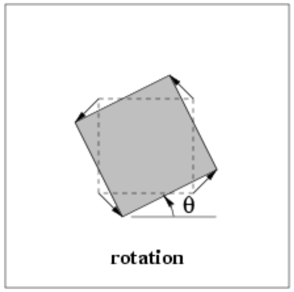
metric structure = structure up to similarity (in literature)

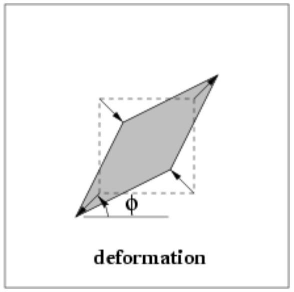
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Class VI: Projective transformations

$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line
(ratio of ratio)

Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon,

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

<u>Example</u>:

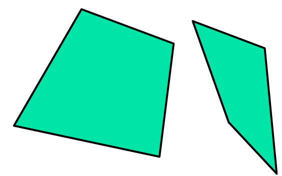
$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Overview transformations

Projective
8dof

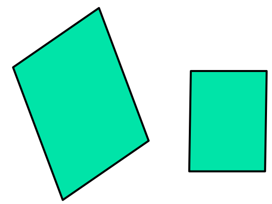
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

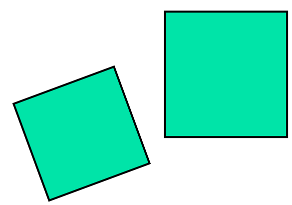
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity $\mathbf{l}_\infty$

Similarity
4dof

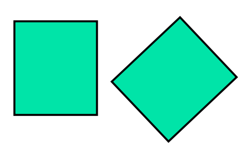
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

Number of invariants?

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

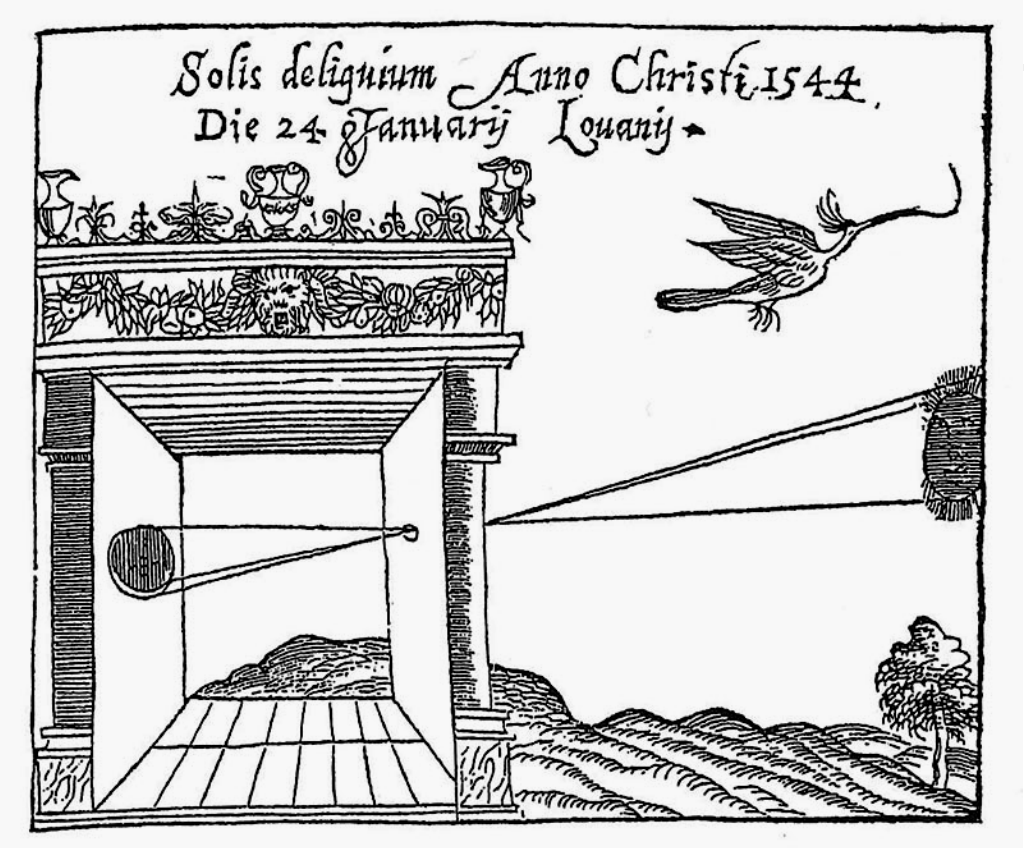
e.g. configuration of 4 points in general position has 8 dof (2/pt)

and so 4 similarity, 2 affinity and zero projective invariants

CAMERA PRINCIPLES

Brief History of Imaging

1544

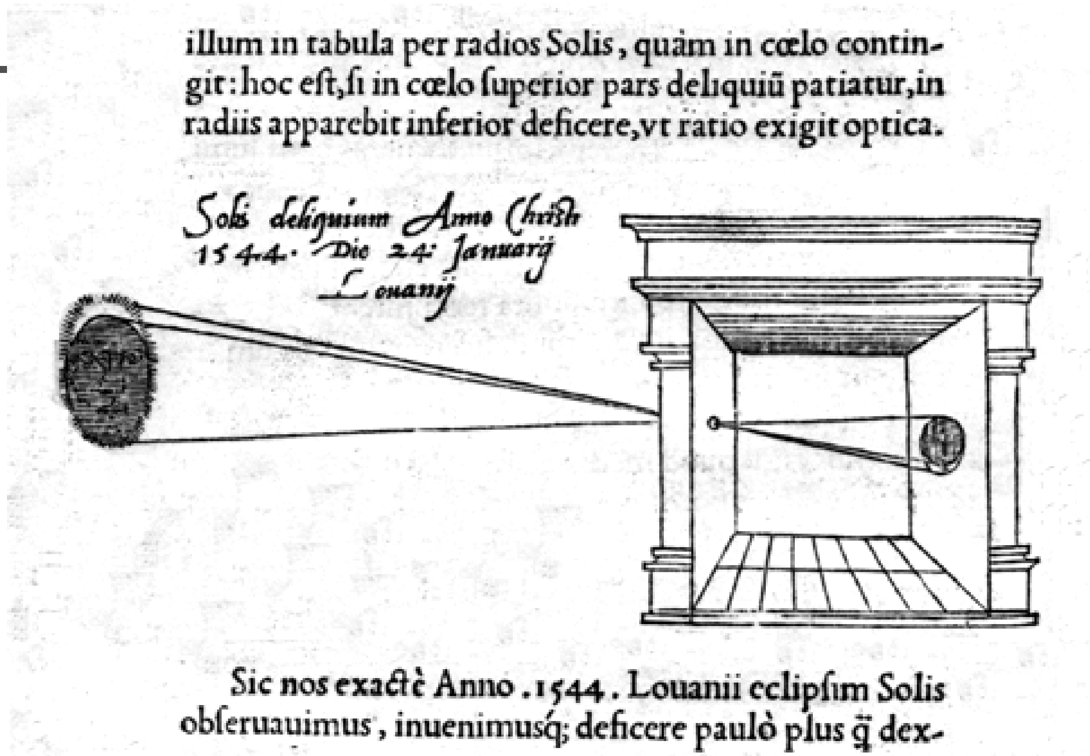


Camera Obscura, Gemma Frisius, 1544

Camera = Latin for "room"
Obscura = Latin for "dark"

Slide courtesy of Shree Nayar

Camera Obscura



illum in tabula per radios Solis, quàm in cœlo contin-
git: hoc eſt, ſi in cœlo ſuperior pars deliquiũ patiatur, in
radiis apparebit inferior deficere, vt ratio exigit optica.

Solis deliquium Anno Christi
1544. Die 24. Januarij
Louanij

Sic nos exactè Anno .1544. Louanii eclipſim Solis
obſeruauimus, inuenimusq; deficere paulò plus q̃ dex-

"When images of illuminated objects ... penetrate through a small hole into a very dark room ... you will see [on the opposite wall] these objects in their proper form and color, reduced in size ... in a reversed position, owing to the intersection of the rays".

Da Vinci

http://www.acmi.net.au/AIC/CAMERA_OBSCURA.html (Russell Naughton)

Jetty at Margate England, 1898.

'Famous Camera Obscura at Santa Monica, Calif'. c.1900

http://brightbytes.com/cosite/collection2.html (Jack and Beverly Wilgus)

Brief History of Imaging

1544

1558

"The Girl with the Red Hat" c. 1665

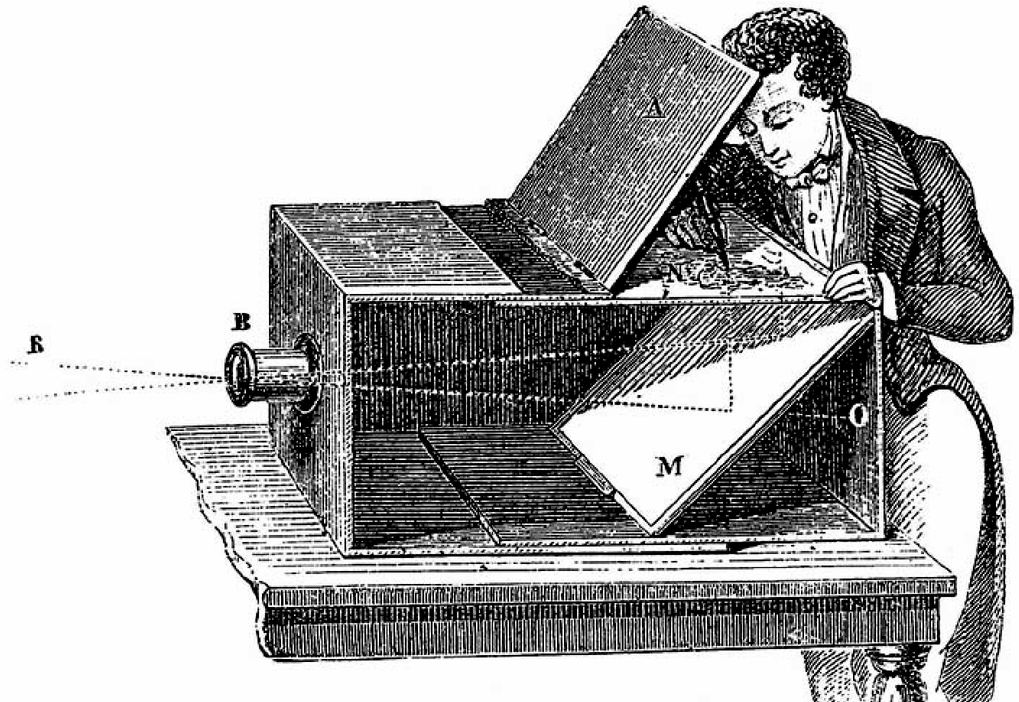


Fig. 434.

Lens Based Camera Obscura, 1558

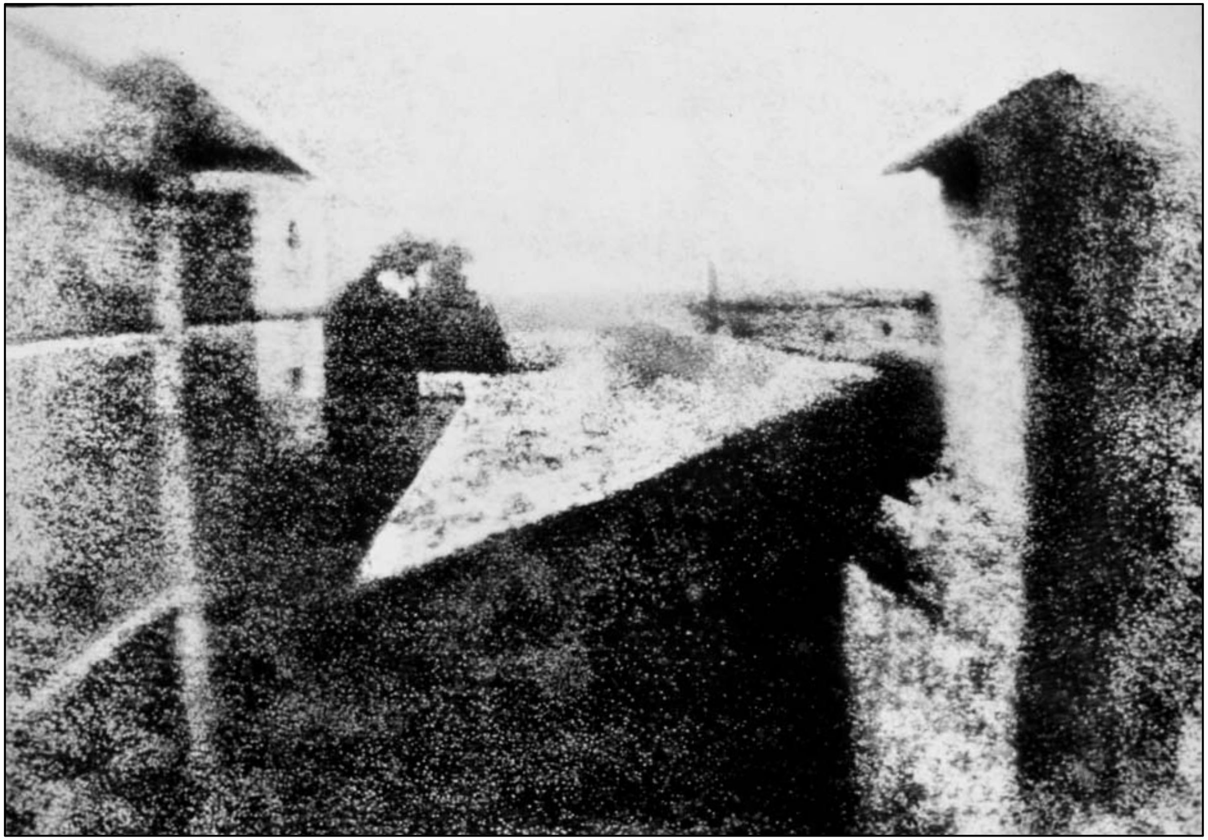
- Used to observe eclipses (eg., Bacon, 1214-1294)
- By artists (eg., Vermeer).

Brief History of Imaging

1544

1558

1827



View from the Window at Le Gras,
Joseph Nicéphore Niépce, 1827

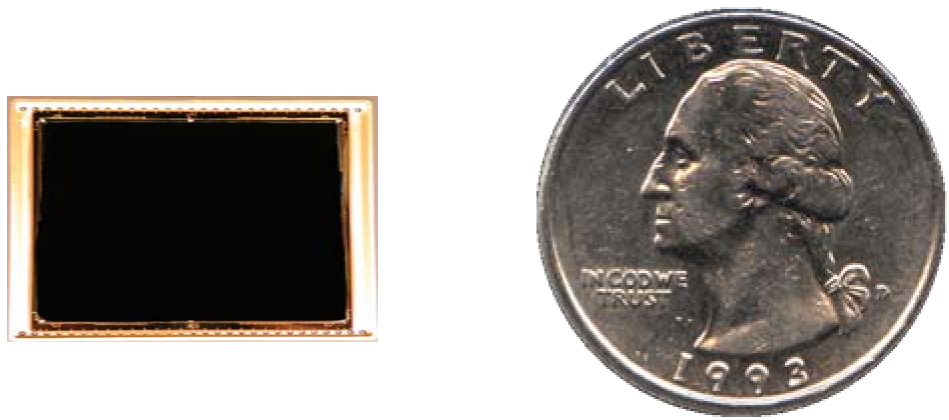
Brief History of Imaging

1544
1558
1827
1839

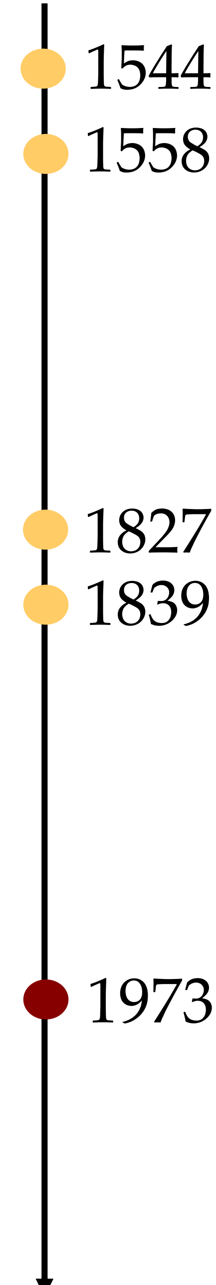
Still Life, Louis Jaques Mande Daguerre, 1839

Slide courtesy of Shree Nayar

Brief History of Imaging



Silicon Image Detector,
Fairchild Semiconductor and Texas Instrument, 1973



Slide courtesy of Shree Nayar

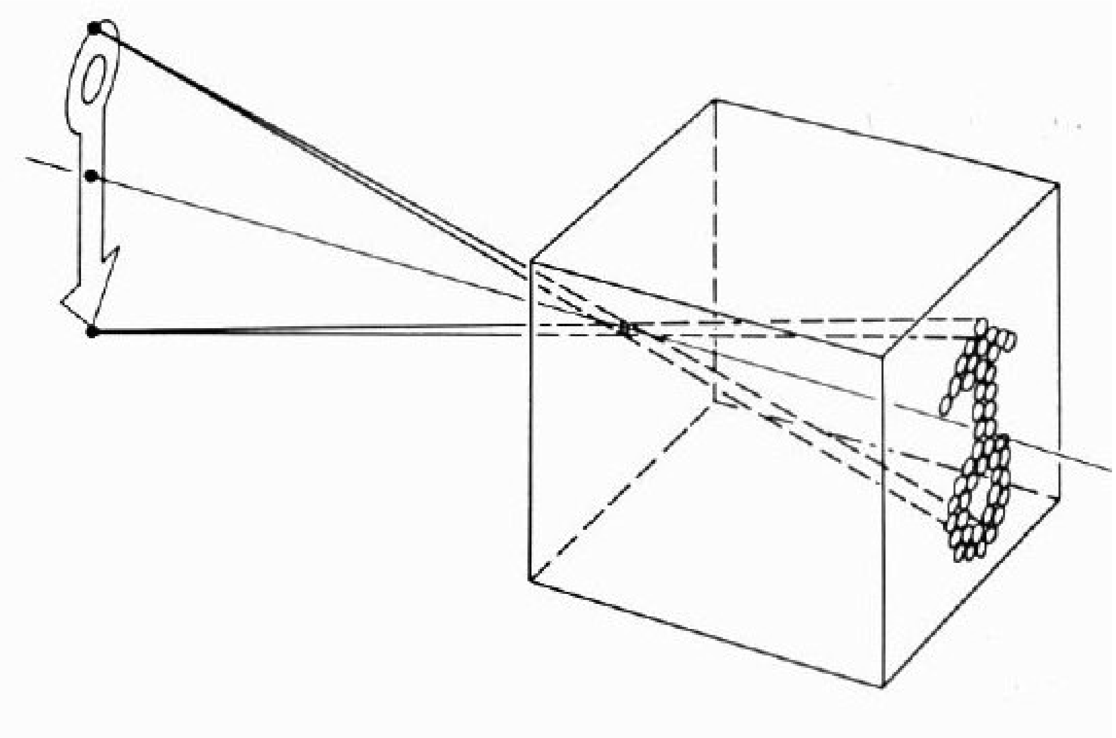
Brief History of Imaging

Digital Cameras

1544
1558
1827
1839
1973
1994

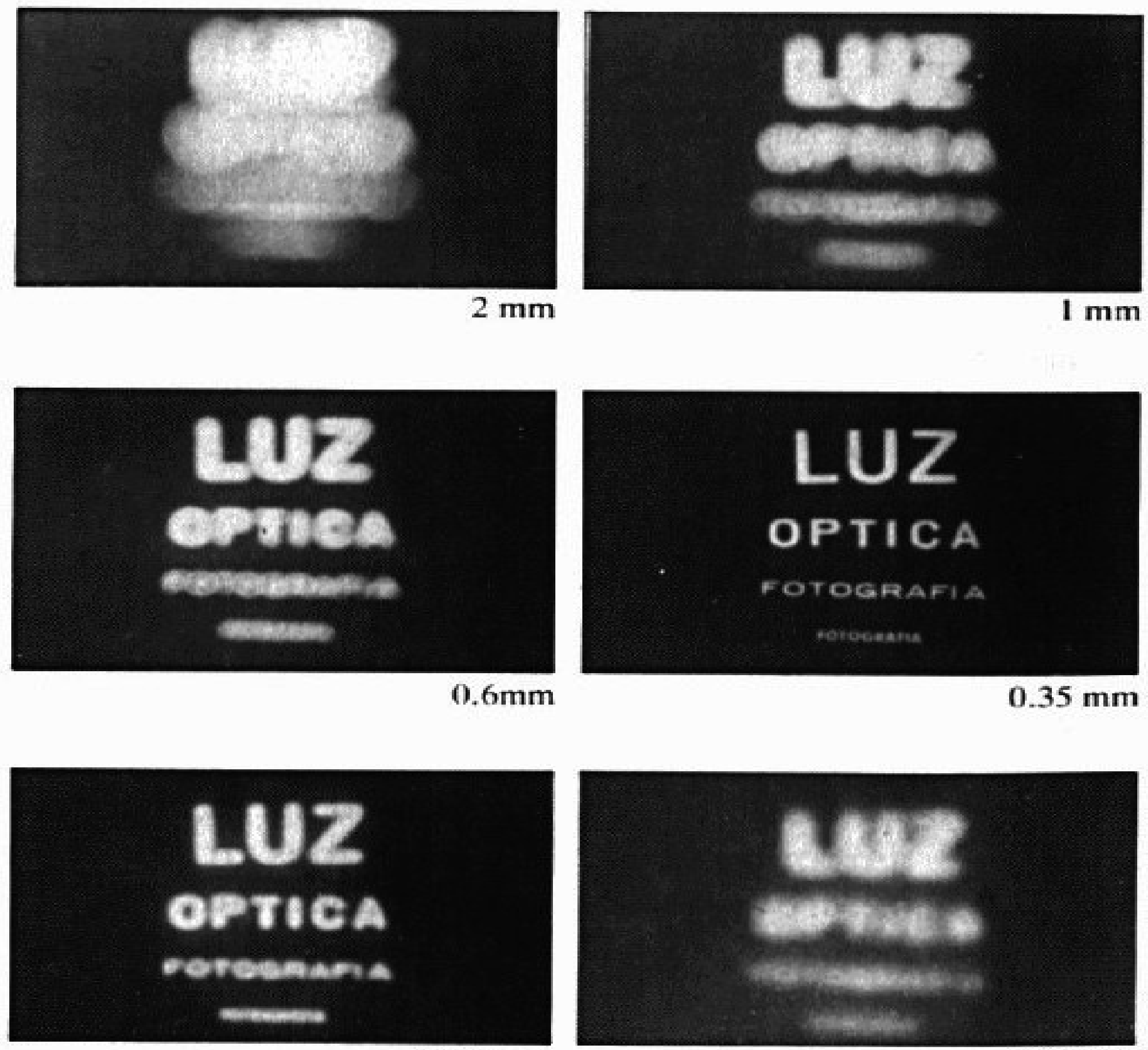
Slide courtesy of Shree Nayar

Pin-hole Camera



- The first camera: Known to Aristotle
- You can make it with a can
- How does the pin-hole (aperture) size affect the image?

Shrinking the aperture

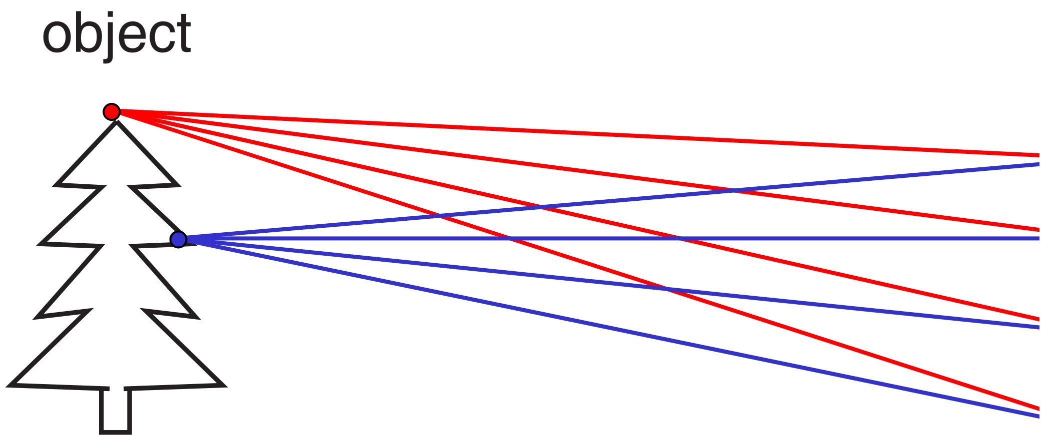


2 mm 1 mm 0.6mm 0.35 mm 0.15 mm 0.07 mm

Why not make the aperture as small as possible?

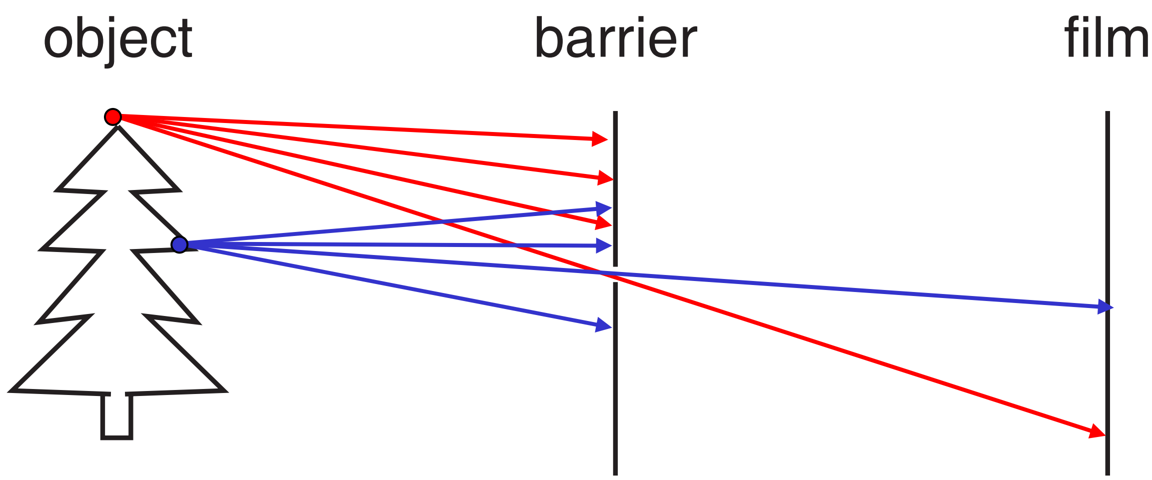
- Less light gets through
- *Diffraction* effects...

Image formation



- Let’s design a camera
  - Idea 1: put a piece of film in front of an object
  - Do we get a reasonable image?

Pinhole camera



- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture(光圈)**
  - How does this transform the image?

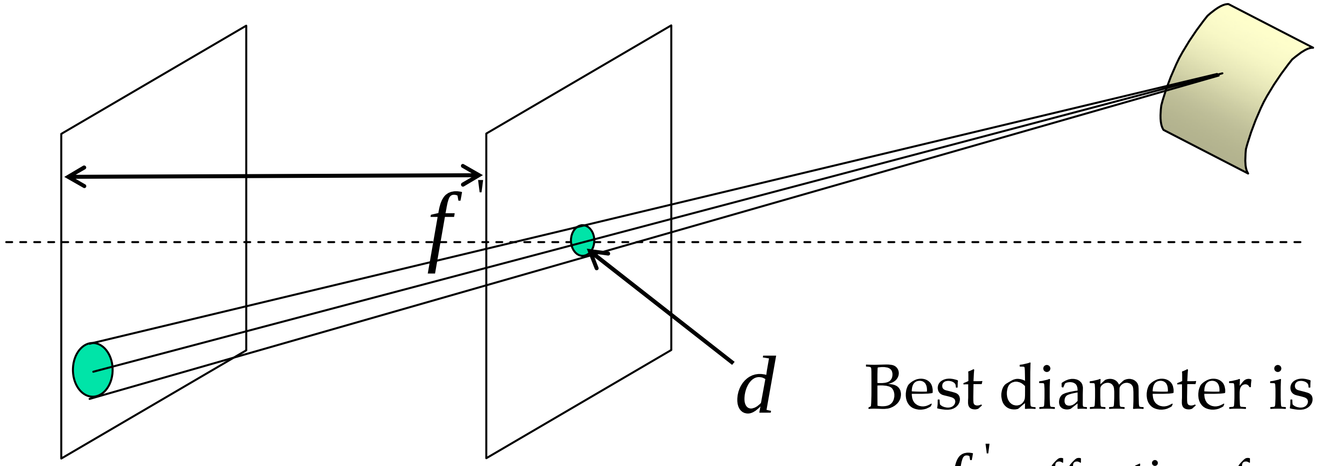
Pin-hole Images

Exposure 4 seconds

Exposure 96 minutes

Limits of Pin-hole Camera

- Aperture has to be small
- The smaller the aperture
  - Less light thus darker images
  - Diffraction（衍射）

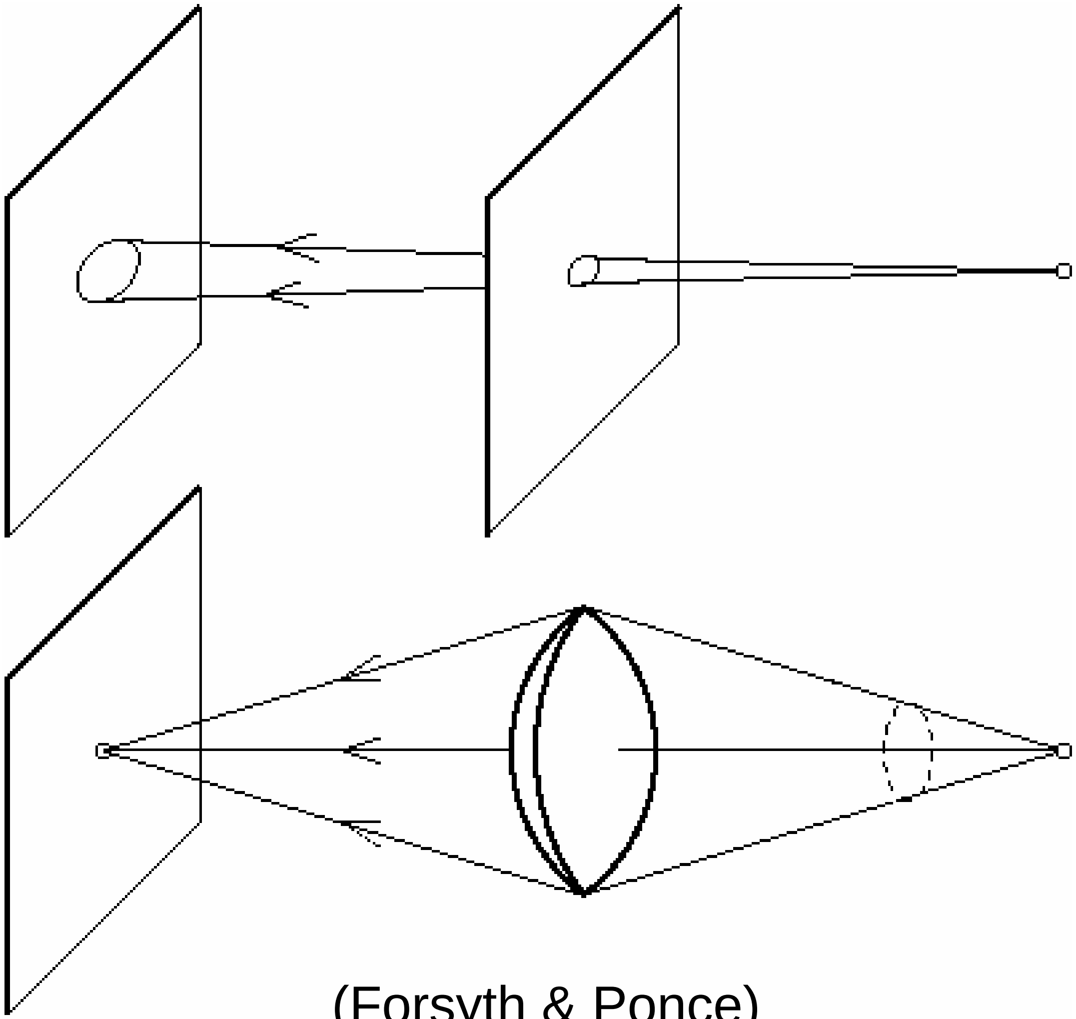


Best diameter is $d = 2\sqrt{f'\lambda}$

f': effective focal length

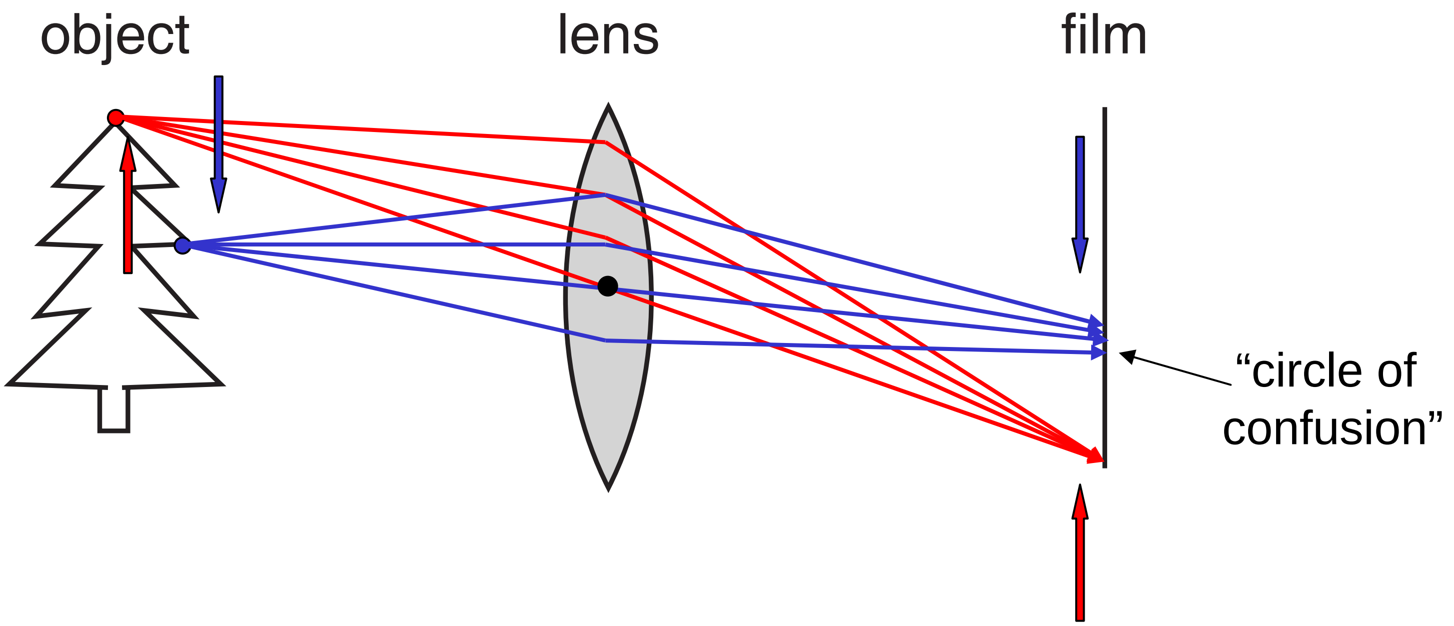
For example, $f' = 50\text{mm}$ $\lambda = 600\text{nm}$ $\longrightarrow$ $d = 0.36\text{mm}$

The reason for lenses



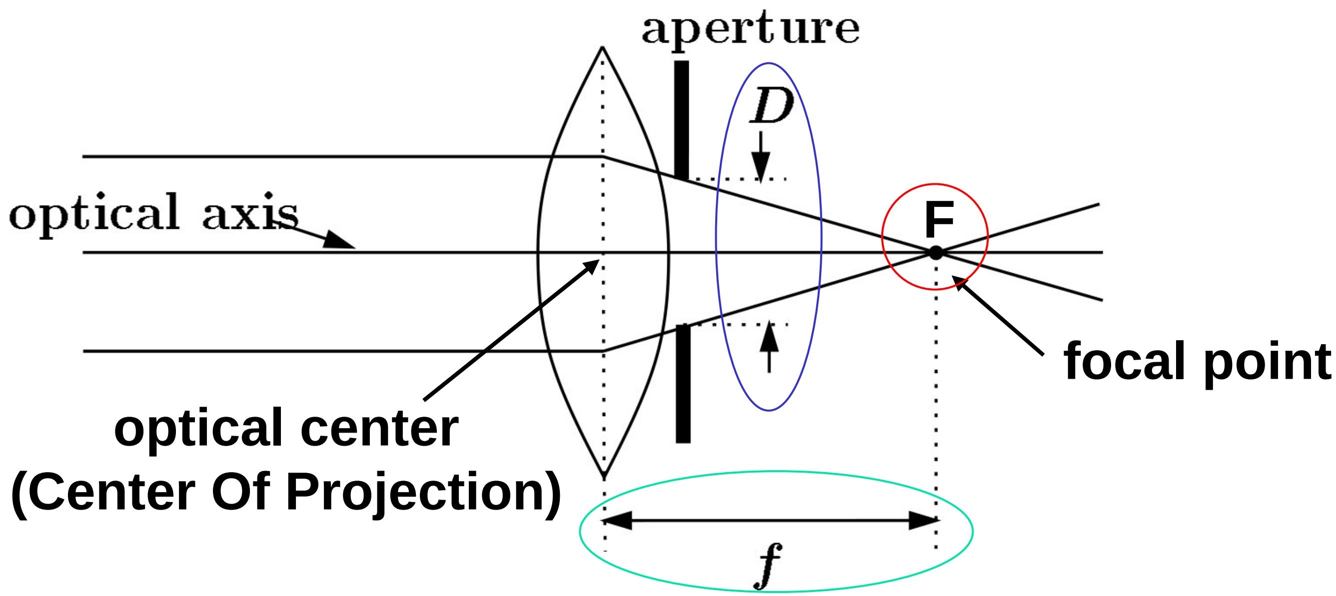
(Forsyth & Ponce)

Adding a lens



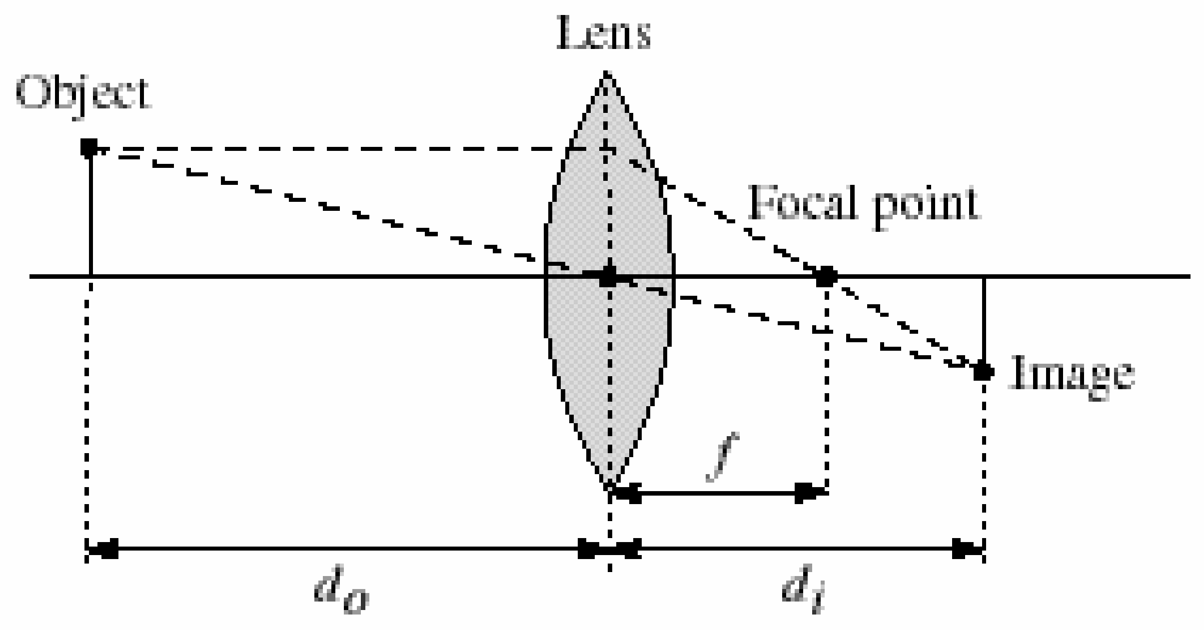
- A lens focuses light onto the film
  - There is a specific distance at which objects are “in focus”
  - other points project to a “circle of confusion” in the image
  - Changing the shape of the lens changes this distance

Lenses



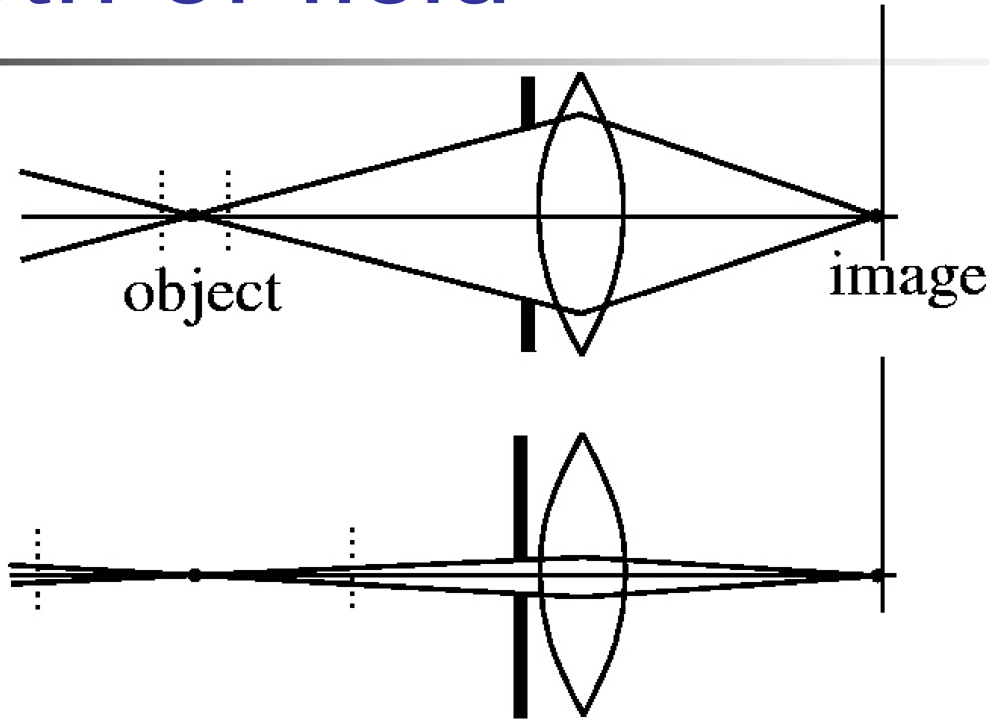
- A lens focuses parallel rays onto a single focal point
  - focal point at a distance *f* beyond the plane of the lens
    - *f* is a function of the shape and index of refraction of the lens
  - Aperture of diameter D restricts the range of rays
    - aperture may be on either side of the lens
  - Lenses are typically spherical (easier to produce)

Thin lenses



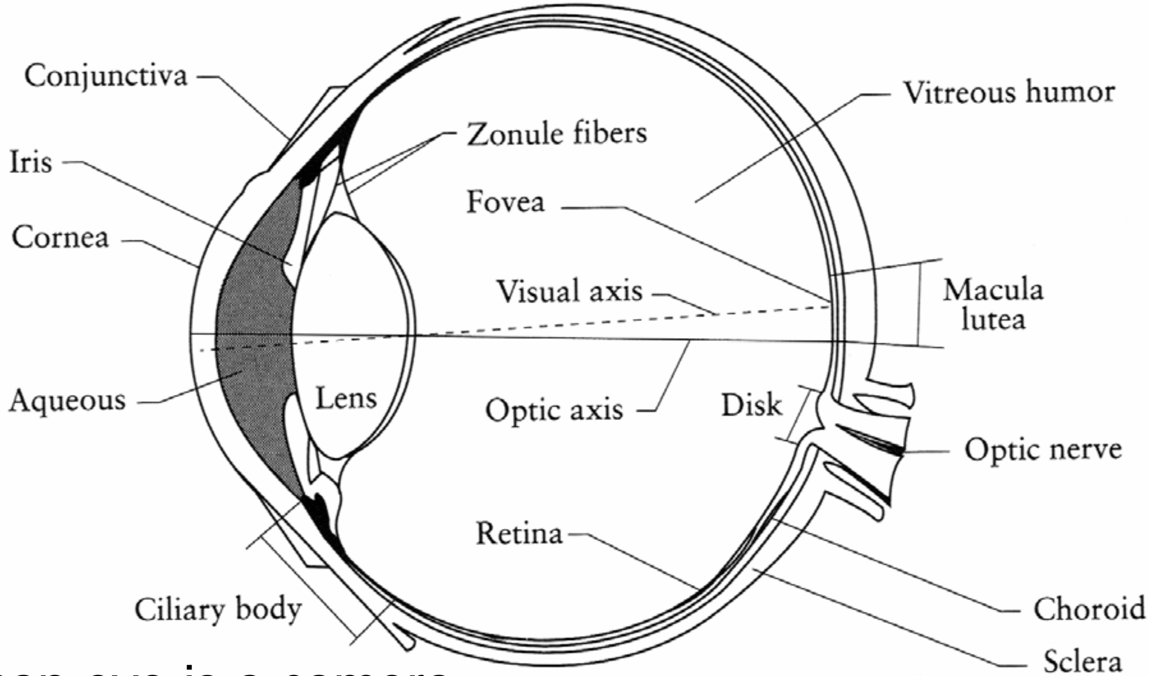
- Thin lens equation: $$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$
  - Any object point satisfying this equation is in focus
  - What is the shape of the focus region?
  - How can we change the focus region?

Depth of field



- Changing the aperture size affects depth of field
  - A smaller aperture increases the range in which the object is approximately in focus

The eye

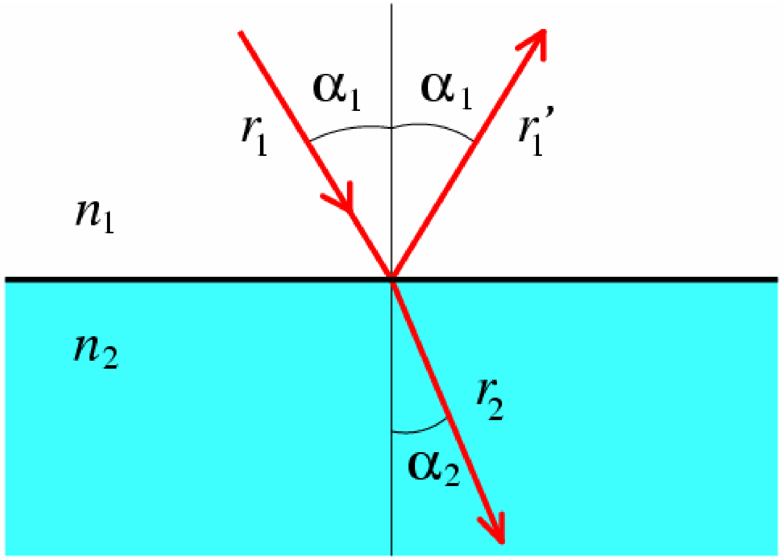


- The human eye is a camera
  - **Iris** - colored annulus with radial muscles
  - **Pupil** - the hole (aperture) whose size is controlled by the iris
  - What's the "film"?
    - photoreceptor cells (rods and cones) in the **retina**

Digital camera

- A digital camera replaces film with a sensor array
  - Each cell in the array is light-sensitive diode（光敏二极管）that converts photons to electrons
  - Two common types
    - Charge Coupled Device (CCD) （电荷耦合元件）
    - CMOS
  - http://electronics.howstuffworks.com/digital-camera.htm

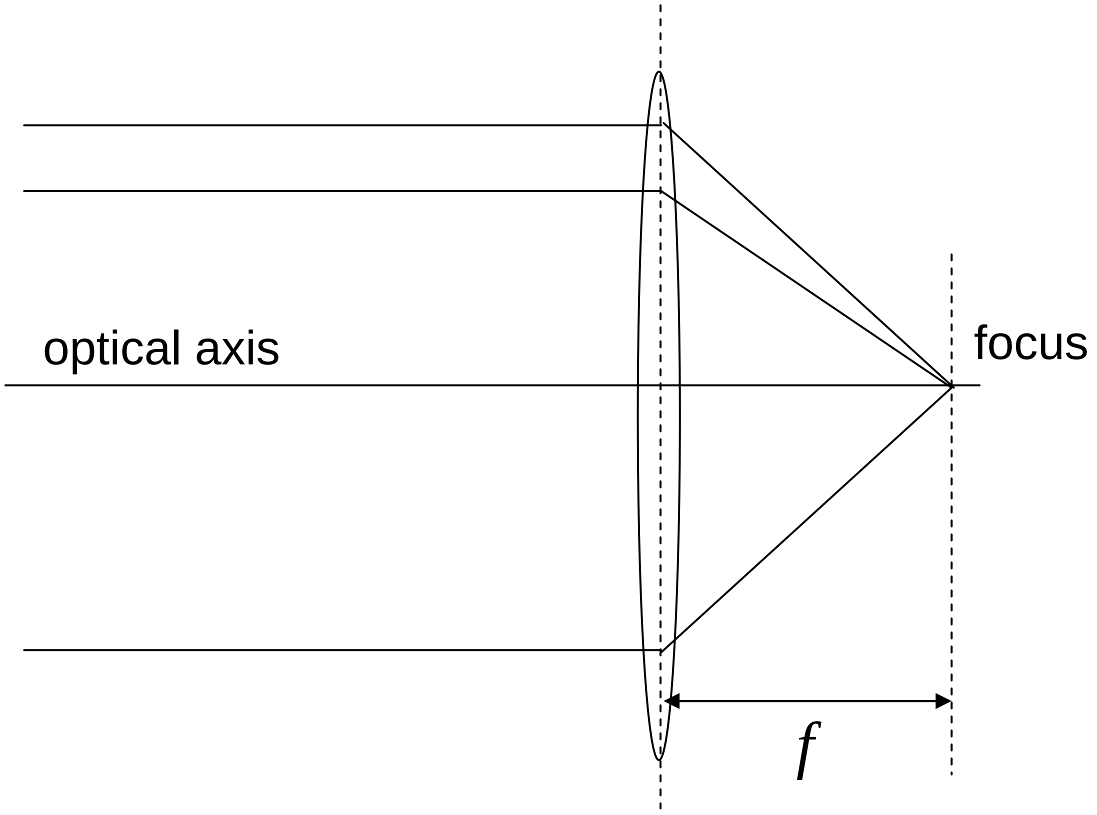
Snell's Law



Snell's law

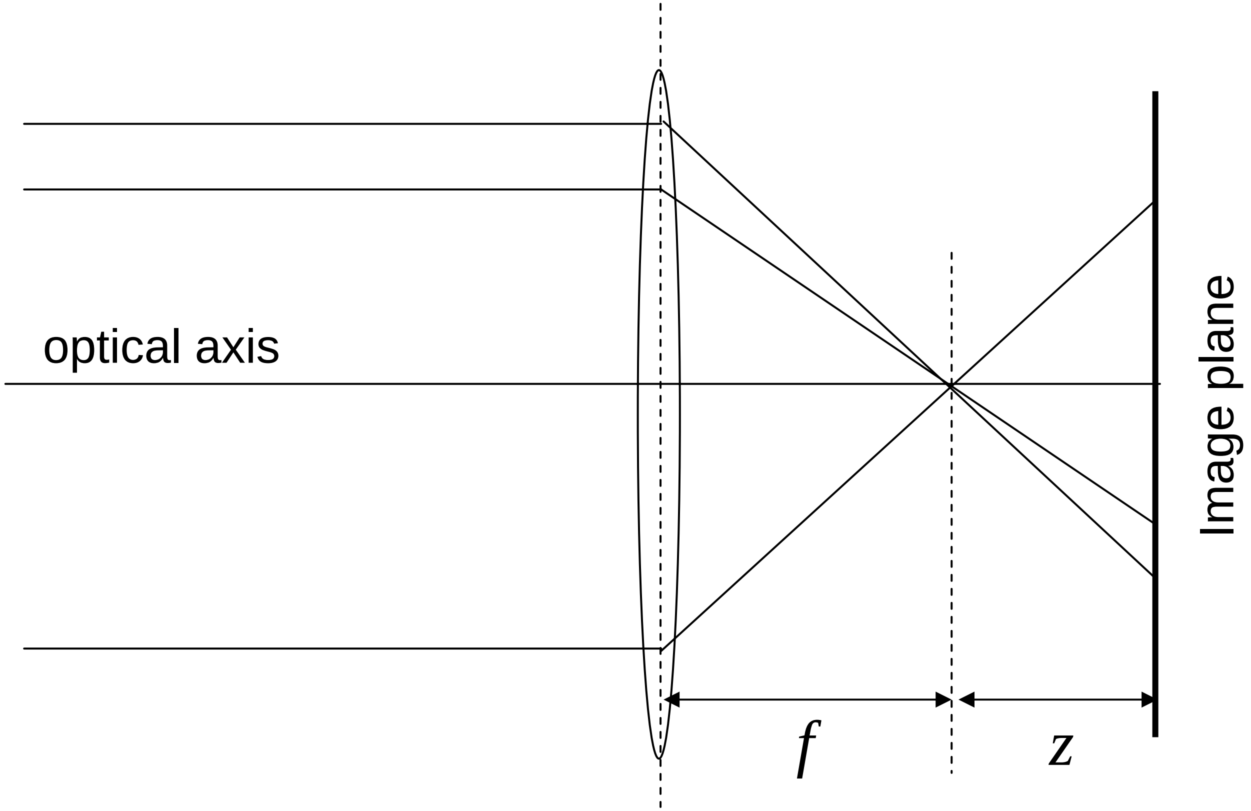
$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2$$

Thin Lens: Definition



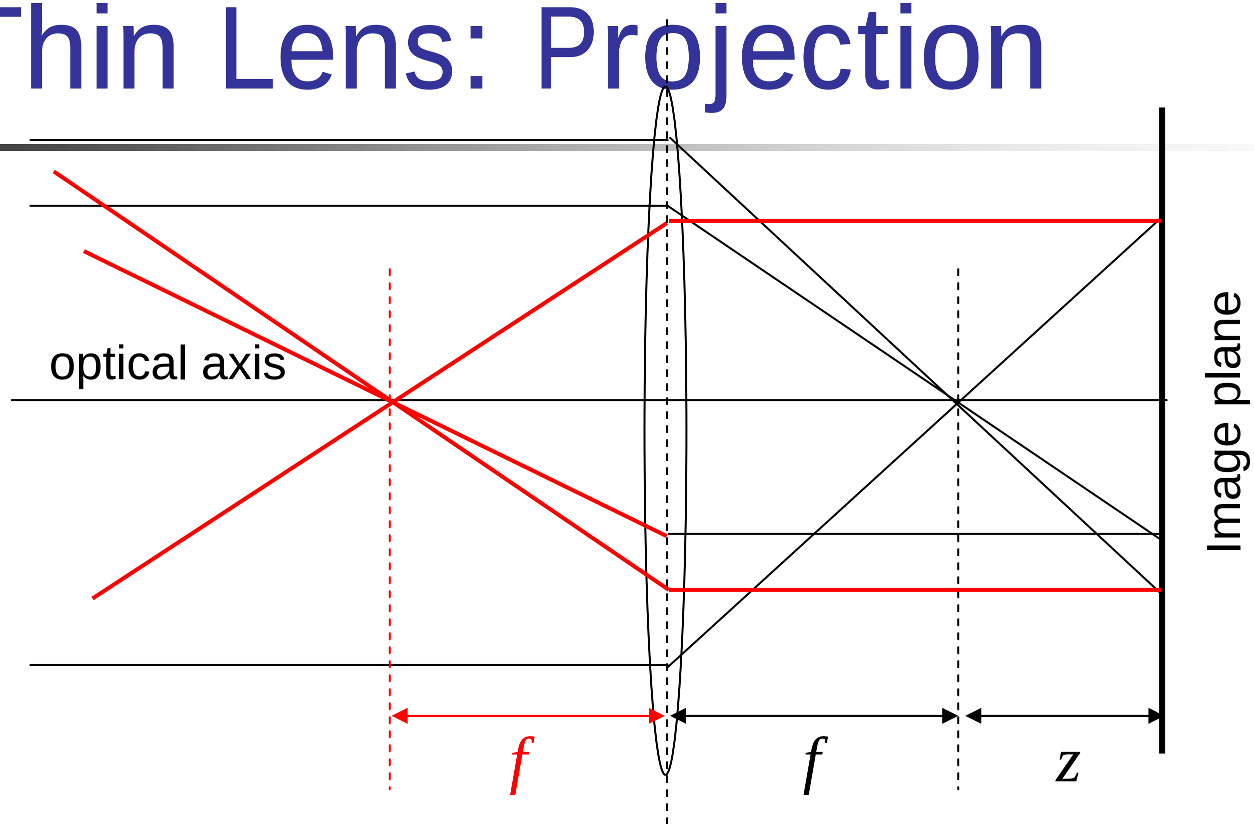
Spherical lense surface: Parallel rays are refracted to single point

Thin Lens: Projection



Spherical lense surface: Parallel rays are refracted to single point

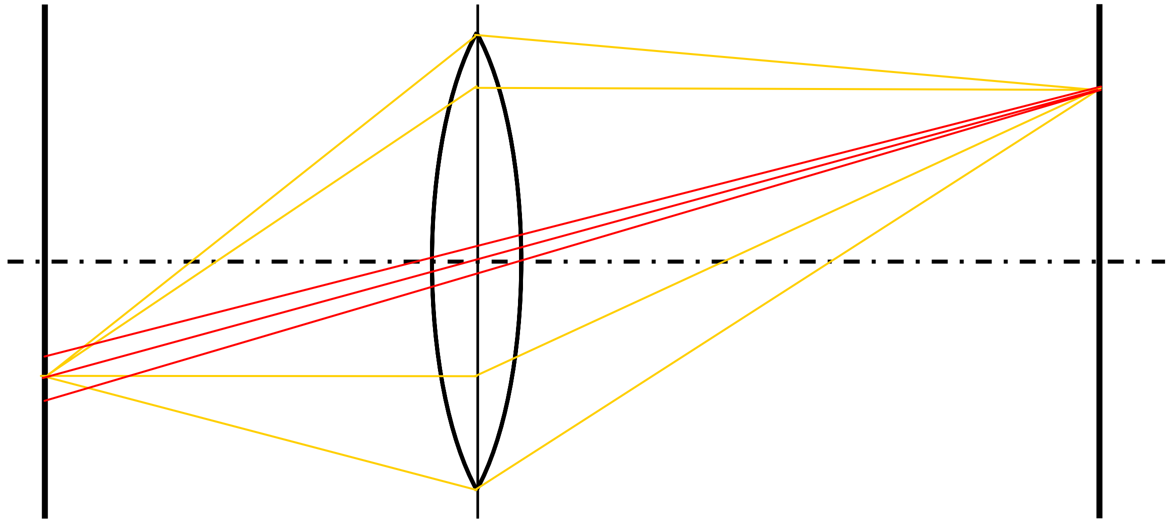
Thin Lens: Projection



Symmetry: Rays passed the focal point are refracted to parallel rays

Lens

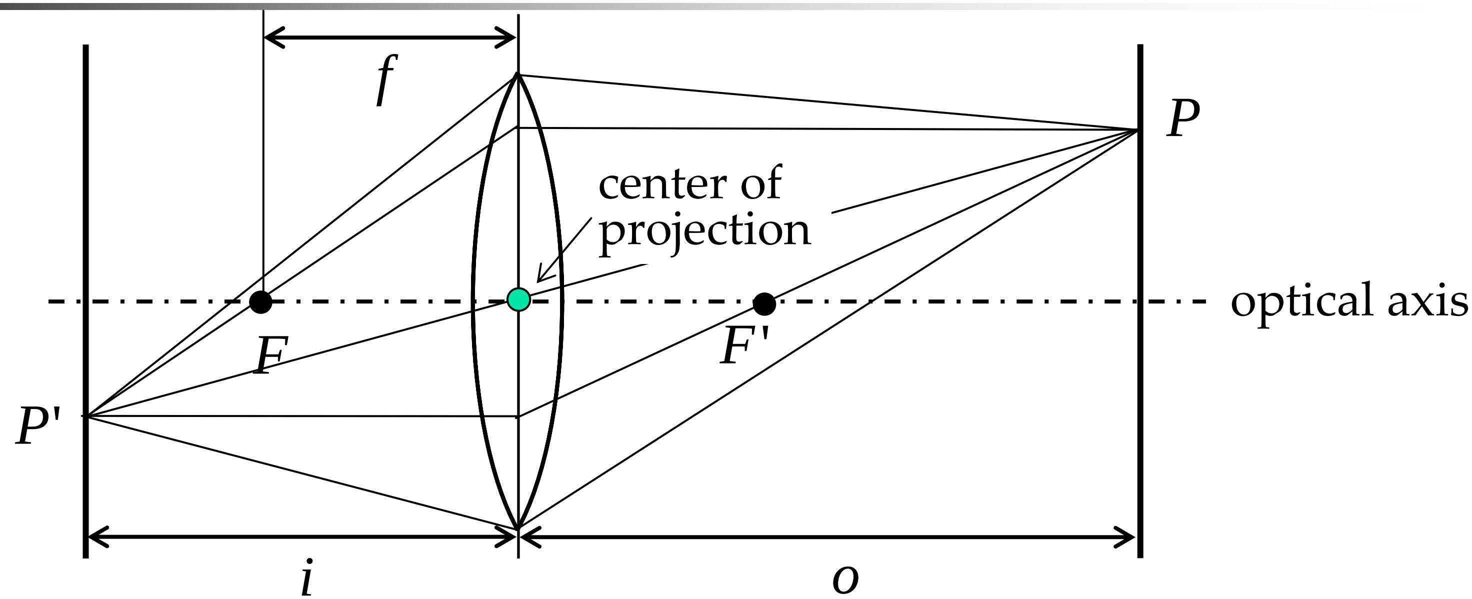
- Ideal lens realizes the same projection as a pinhole but **gathers much more light**!



Thin Lens: Properties

1. Any ray entering a thin lens parallel to the optical axis must go through the focus on other side
2. Any ray entering through the focus on one side will be parallel to the optical axis on the other side
3. Any ray passing through the optical center does not change its direction

Thin-lens Law (Gaussian Lens Formula)



f : focal length

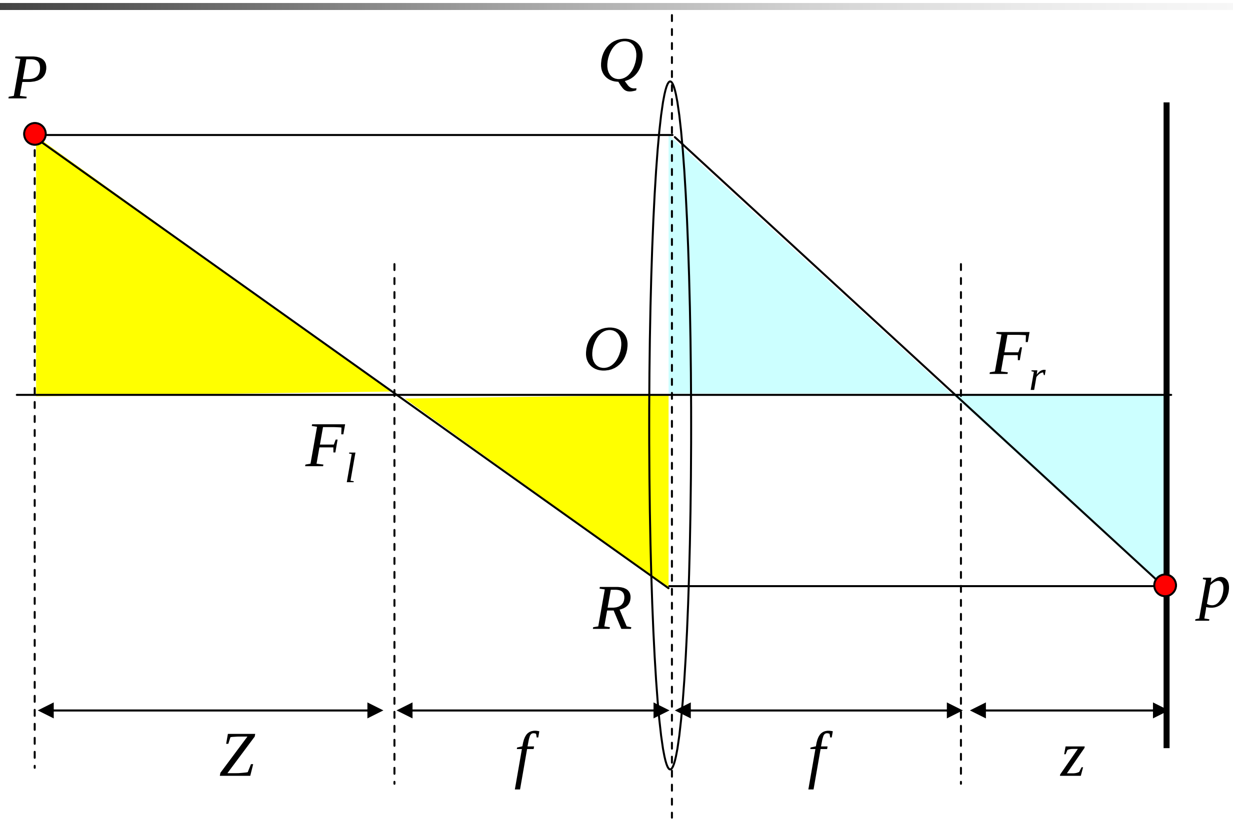
$$\boxed{\frac{1}{i}+\frac{1}{o}=\frac{1}{f}}$$

For example, $f=50\text{mm}$ $o=300\text{mm}$ $\longrightarrow$ $i=60\text{mm}$

Thin Lens: Model

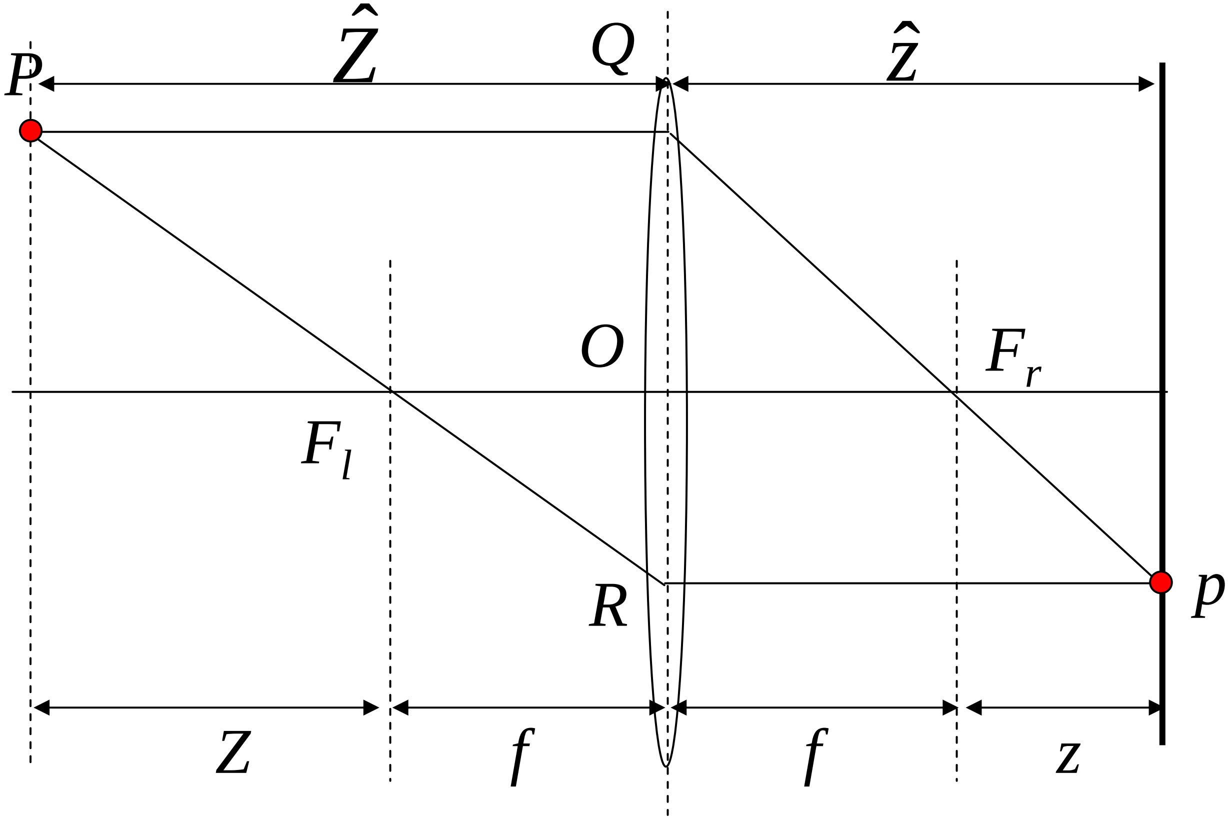
$$\frac{QO}{Z} = \frac{OR}{f}$$

$$\frac{OR}{z} = \frac{QO}{f}$$



$$OR = \frac{f}{Z} QO = \frac{z}{f} QO \qquad \frac{f}{Z} = \frac{z}{f} \Rightarrow f^2 = zZ$$

The Thin Lens Law



$$f^2 = zZ$$

$$\hat{z} = z + f$$

$$\hat{Z} = Z + f$$

$$\frac{1}{\hat{z}} + \frac{1}{\hat{Z}} = \frac{1}{f}$$

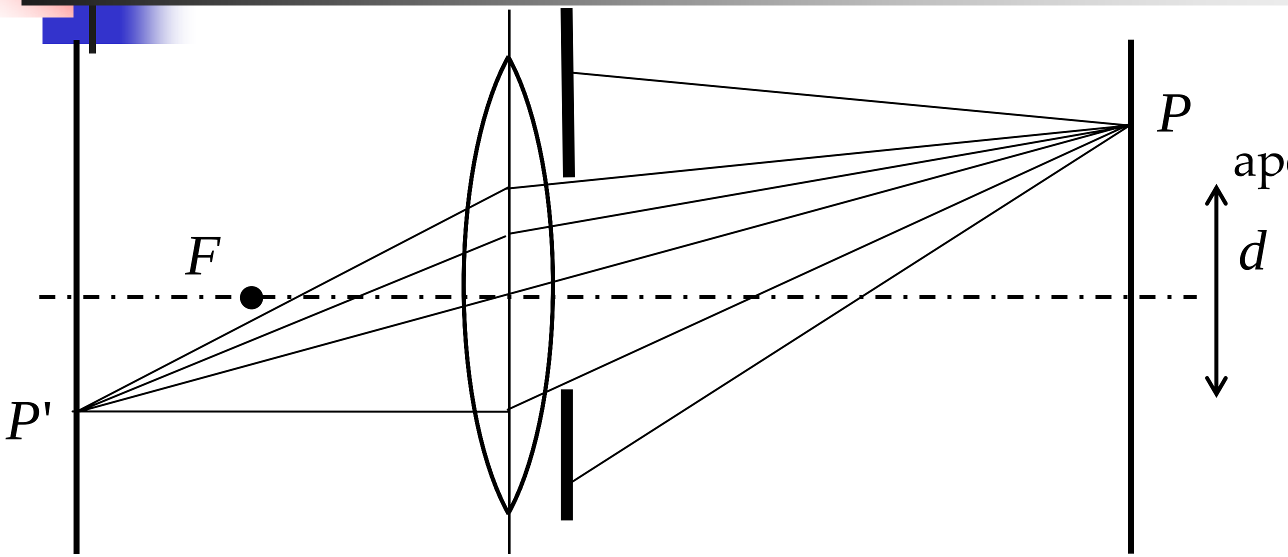
Limits of the Thin Lens Model

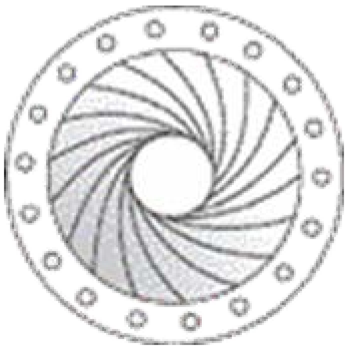
3 assumptions :

1. all rays from a point are focused onto 1 image point
   - Remember thin lens small angle assumption
2. all image points in a single plane
3. magnification $m = \frac{f'}{z_0}$ is constant

Deviations from this ideal are *aberrations*

Aperture and F/stop





iris diaphragm

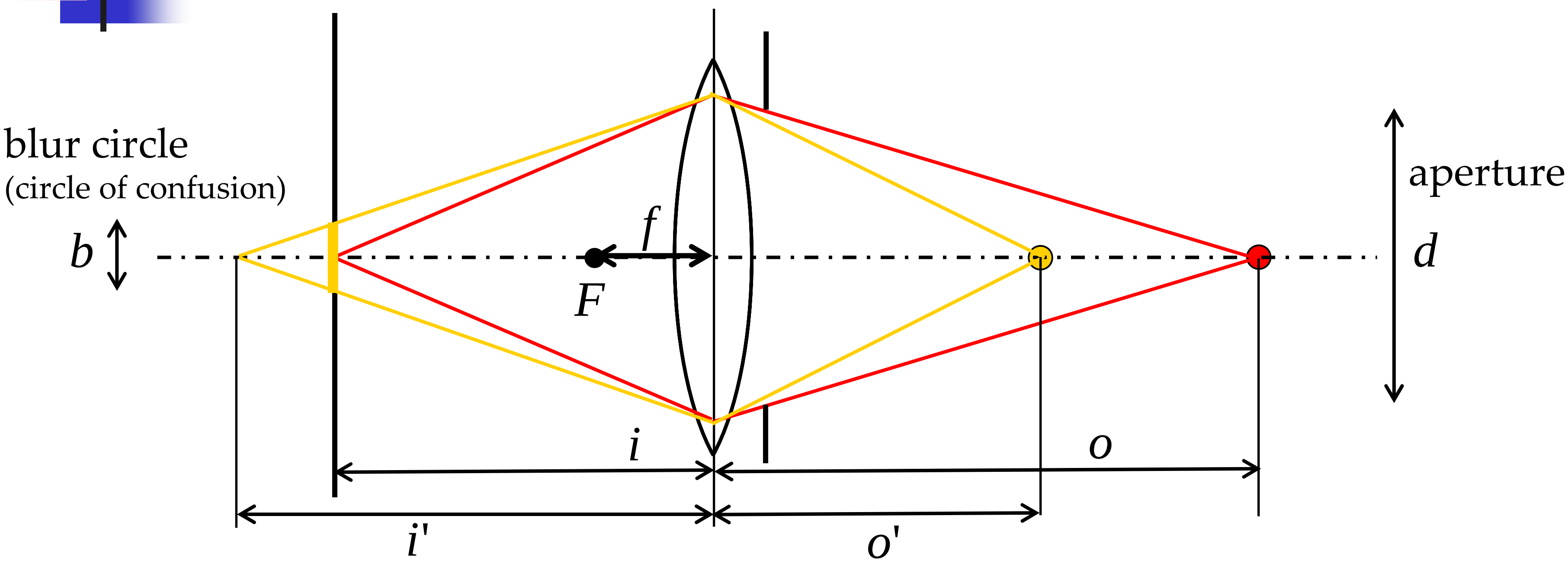
F/stop: for instance, f/1.0 f/1.4 f/2.0 f/2.8 f/4 f/5.6 f/8 f/11

less light

aperture areas is halved at each stop

f/stop =	f/1.4	f/2.0	f/2.8	f/4	f/5.6	f/8	f/11
shutter speed =	1/1000	1/500	1/250	1/125	1/60	1/30	1/15

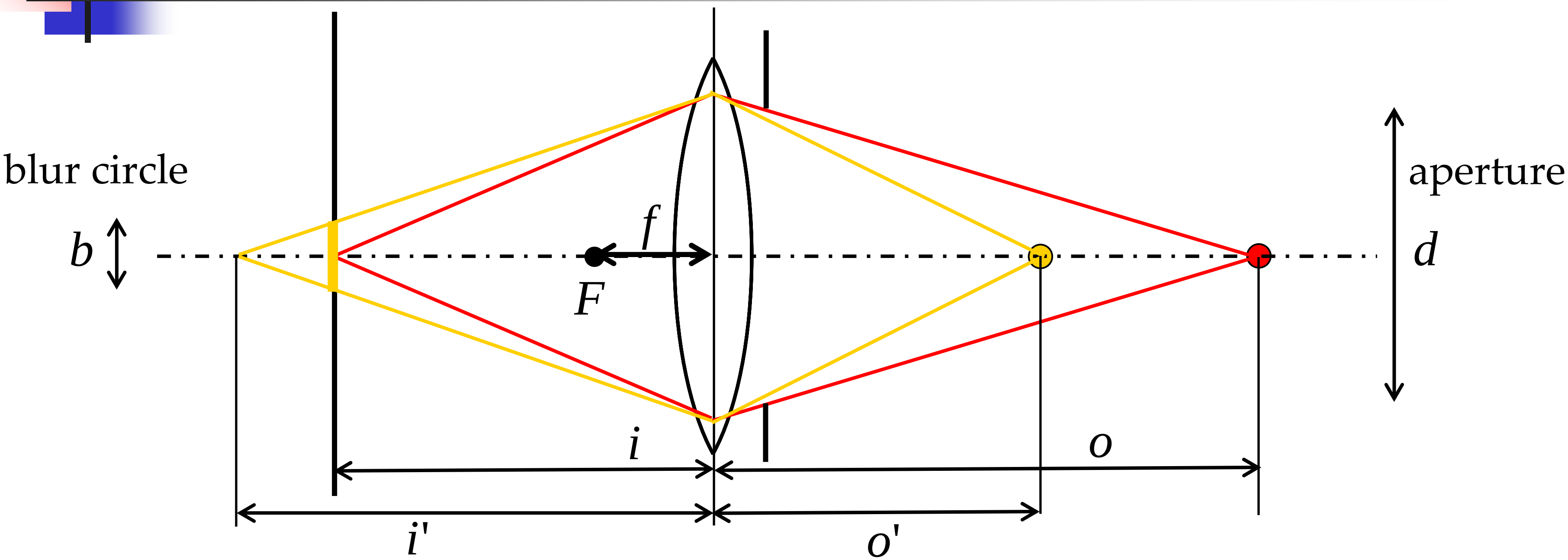
Defocus



$$\frac{1}{i}+\frac{1}{o}=\frac{1}{f},\quad \frac{1}{i'}+\frac{1}{o'}=\frac{1}{f'}\quad \Rightarrow \quad i=\frac{of}{o-f},\quad i'=\frac{o'f}{o'-f}$$

$$\Rightarrow (i'-i)=\frac{f}{(o'-f)}\frac{f}{(o-f)}(o-o') \quad \text{and} \quad b=\frac{d}{i'}|i'-i|$$

Depth of Field



$$b = \frac{d}{i'}\left|\frac{f}{(o'-f)}\frac{f}{(o-f)}(o-o')\right|$$

Range of object distance over which image is sufficiently well focused
i.e. **Range (*o-o′*) for which *b* is less than a pixel of the imaging sensor**

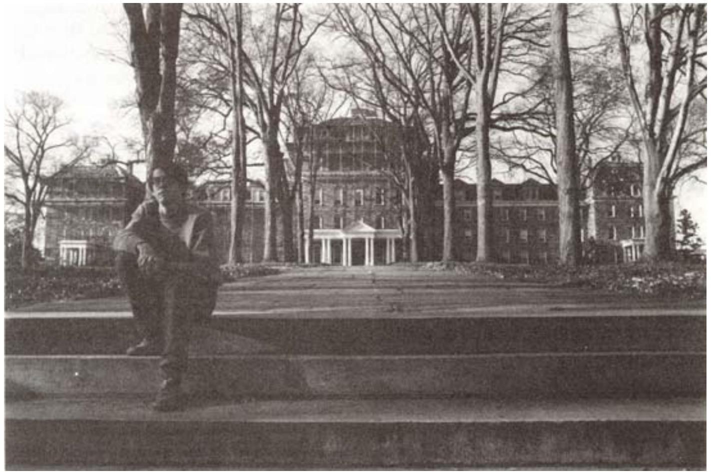
Depth of Field

Changing aperture

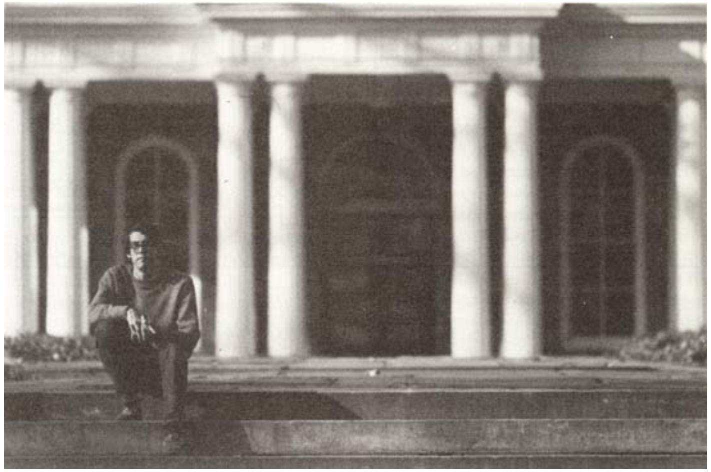
f/11 1/30sec

f/2.8 1/500sec

Changing focal length



wide-angle lens (short f)

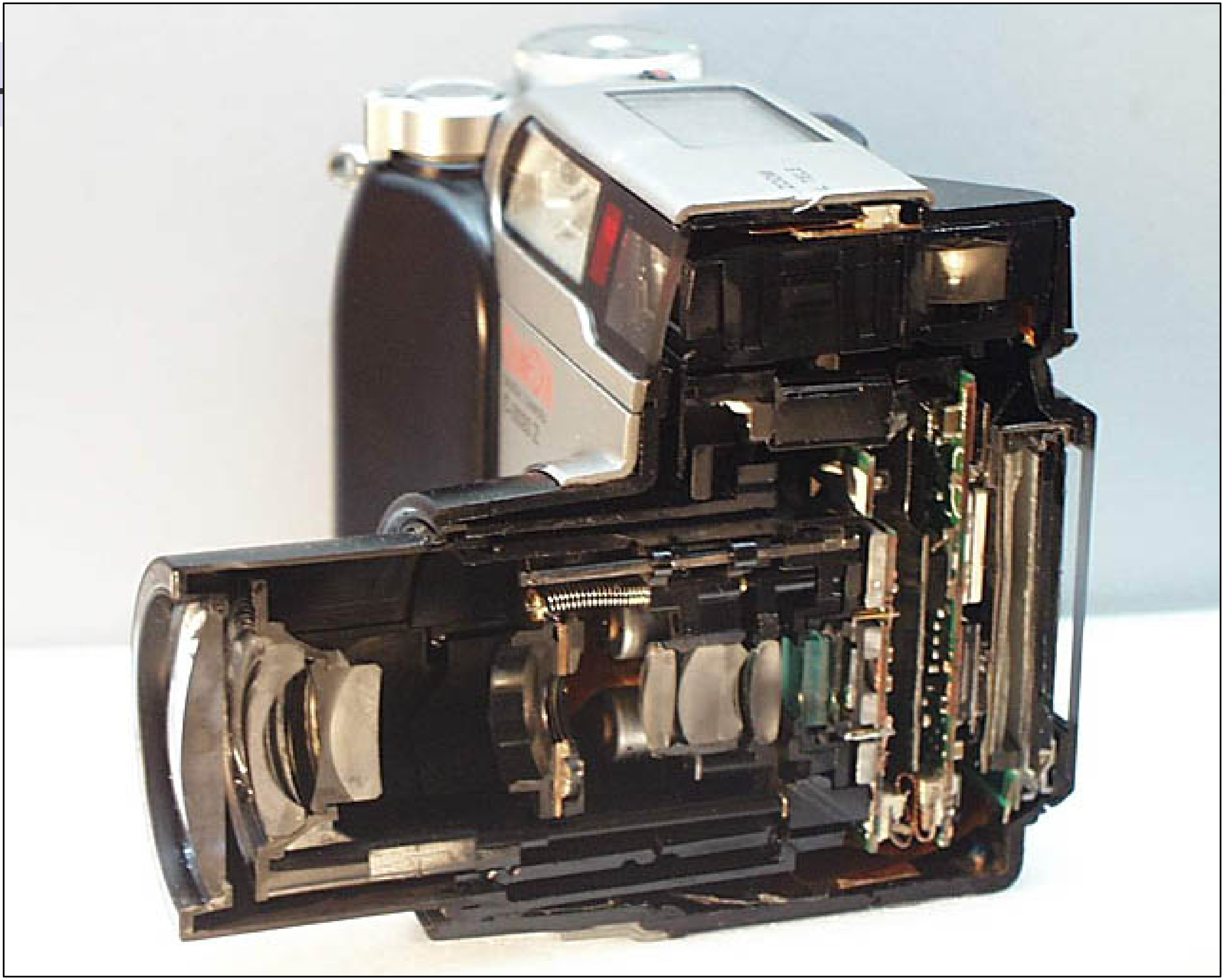


tele-photo lens (long f)

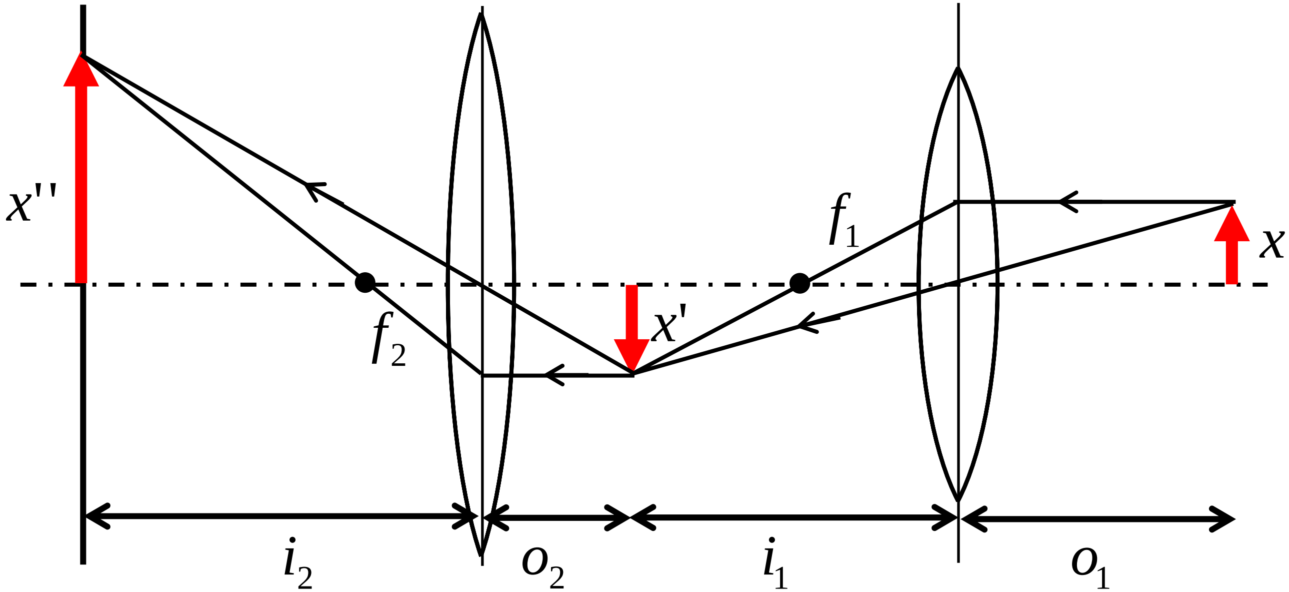
Assumptions for thin lens equation

- Lens surfaces are spherical
- Incoming light rays make a small angle with the optical axis
- The lens thickness is small compared to the radii of curvature
- The refractive index is the same for the media on both sides of the lens

Camera with Lens



Two Lenses



Magnification $$m = \frac{x''}{x} = \frac{i_2}{o_2}\frac{i_1}{o_1}$$

By moving the lenses back and forth, we can **zoom** without moving the object or the image planes.

Distortion

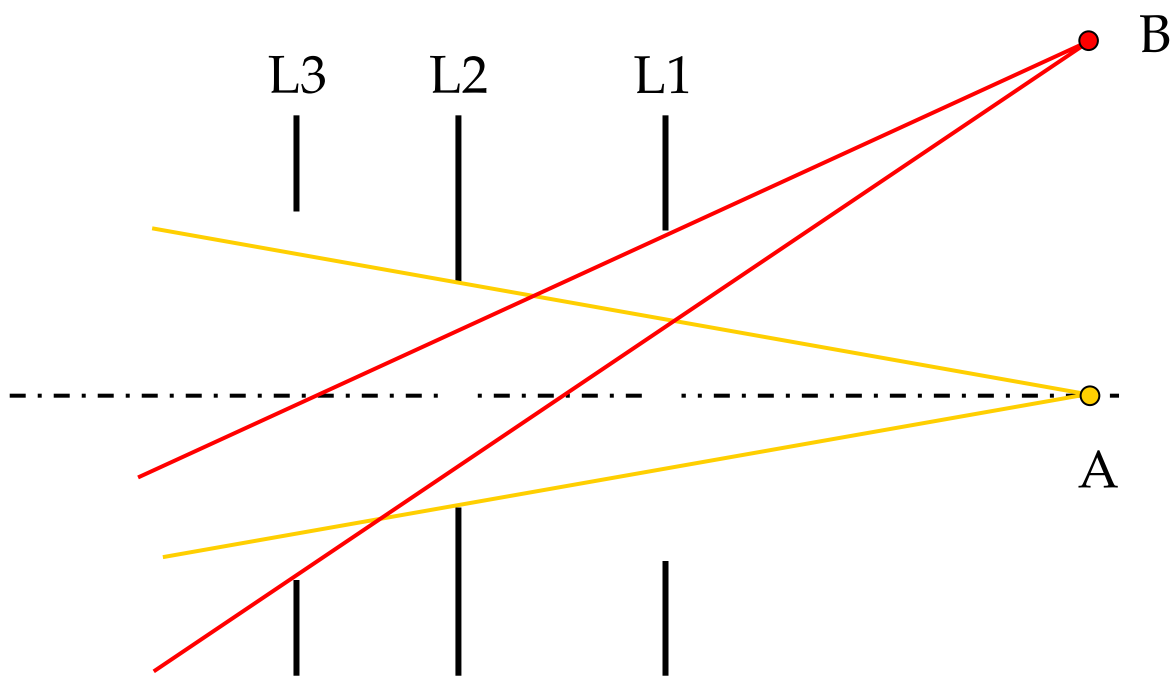
- Radiometric
  - Vignetting(渐晕)
  - Chromatic aberration(光行差)
- Geometric
  - Radial（径向畸变）
  - Tangential（切向畸变）

Vignetting

Vignetting

Optical Vignetting - Aperture dependency

At wider aperture, on the edge of the field, the entrance pupil can be partially shielded by the lens body. This is why optical vignetting increases with aperture.

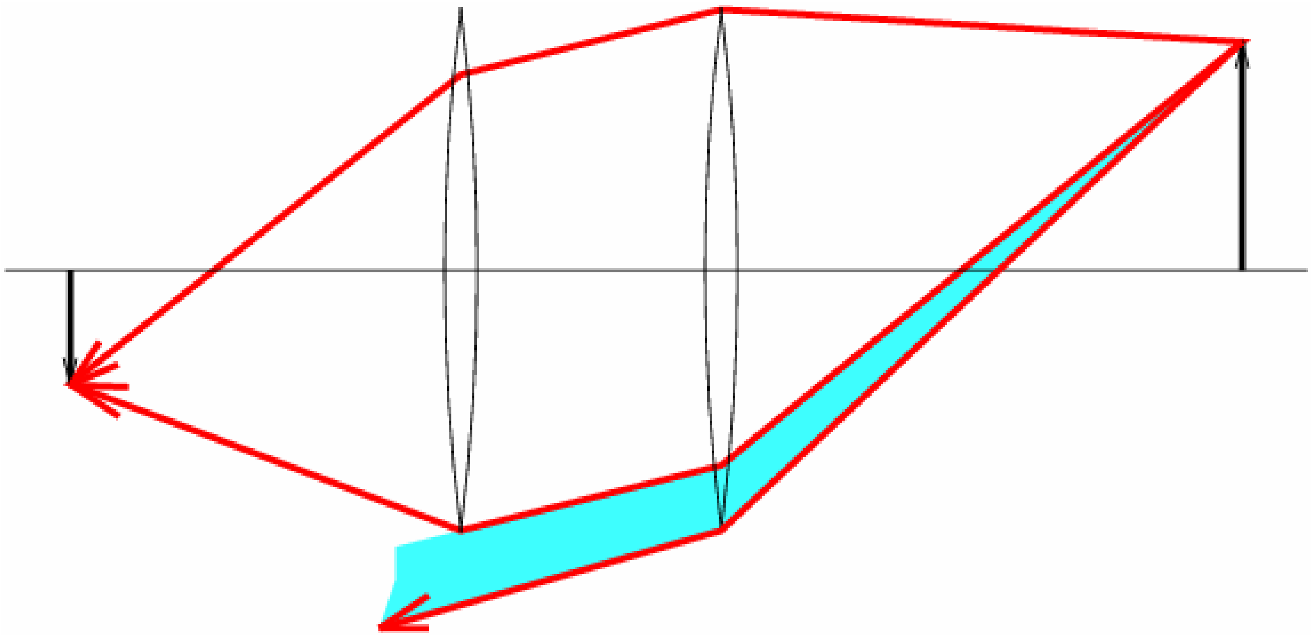


More light passes through lens L3 for scene point A than scene point B.
Results in spatially non-uniform brightness (in the periphery of the image)

Vignetting

Natural Vignetting - Lens Dependency

- •Natural vignetting is inherent to lens design, regardless of aperture.
- •With a zoom lens, it generally increases as the focal length decreases.



Effect: Darkens pixels near the image boundary

Vignetting

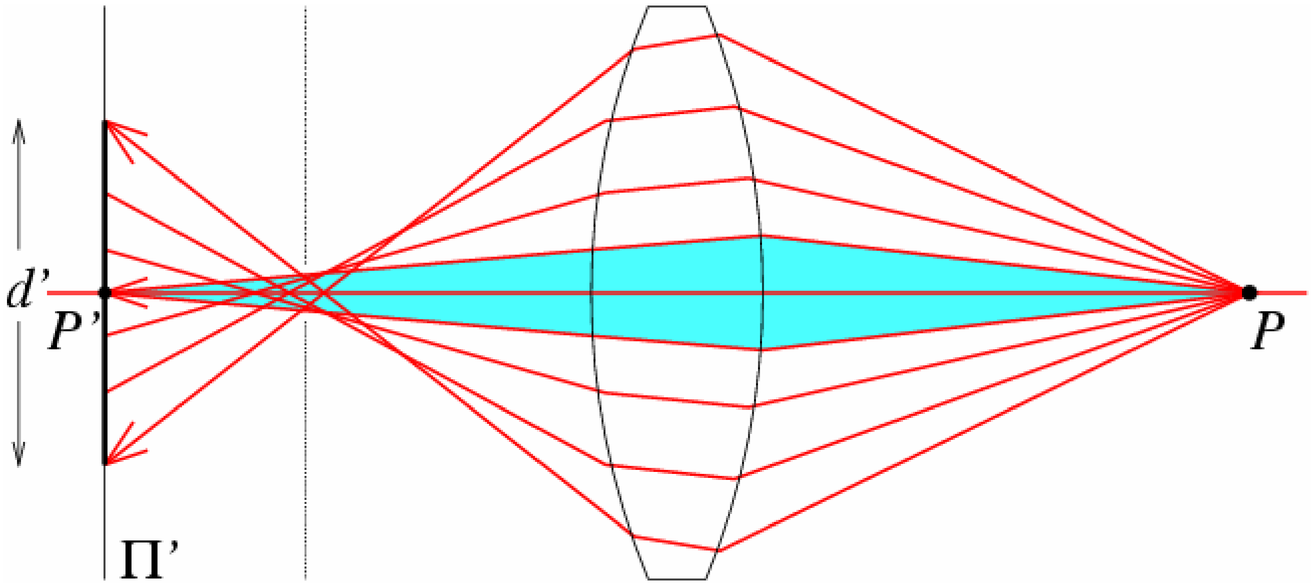
photo by Robert Johnes

Vignetting Correction

Spherical Aberration

Rays parallel to the axis do not converge

Outer portions of the lens yield smaller focal lengths



Chromatic Aberration

rays of different wavelengths focused in different planes

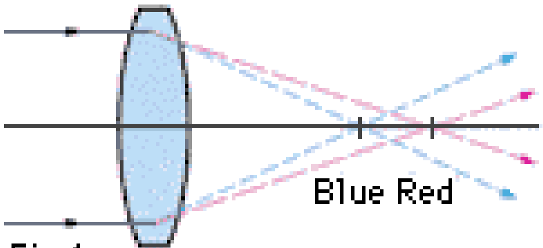


Fig.1
Axial chromatic aderration

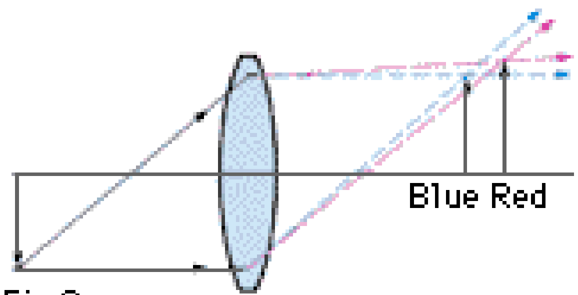
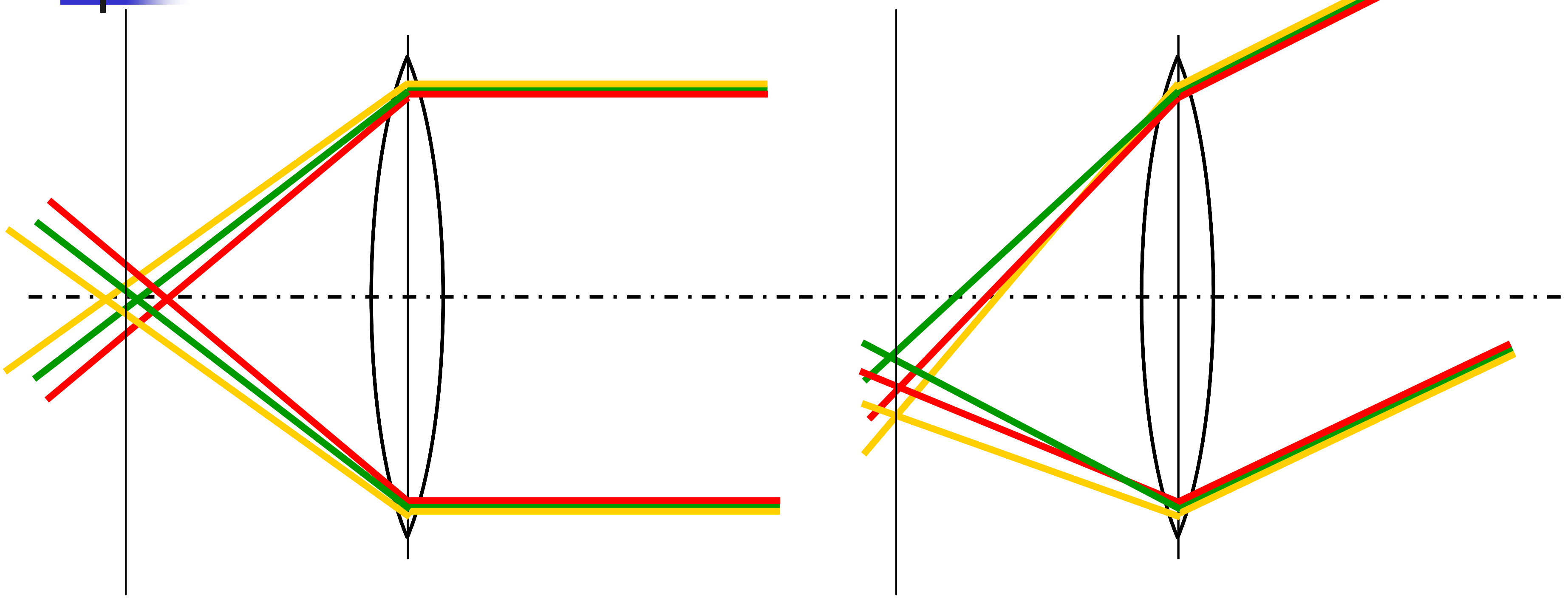


Fig.2
Magnification chromatic aderration

cannot be removed completely

The image is blurred and appears colored at the fringe.

Chromatic Aberration

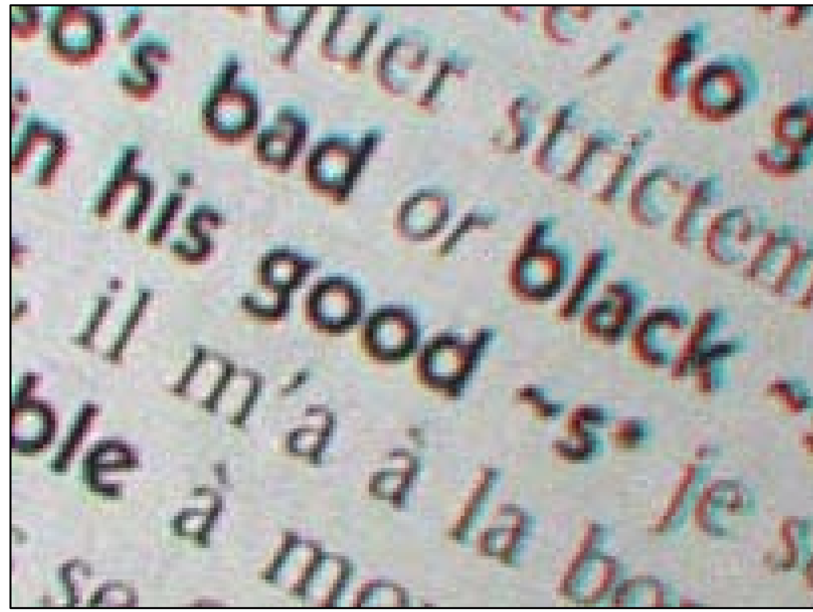


longitudinal chromatic aberration
(axial)

transverse chromatic aberration
(lateral)

Chromatic Aberration

longitudinal chromatic aberration
(axial)



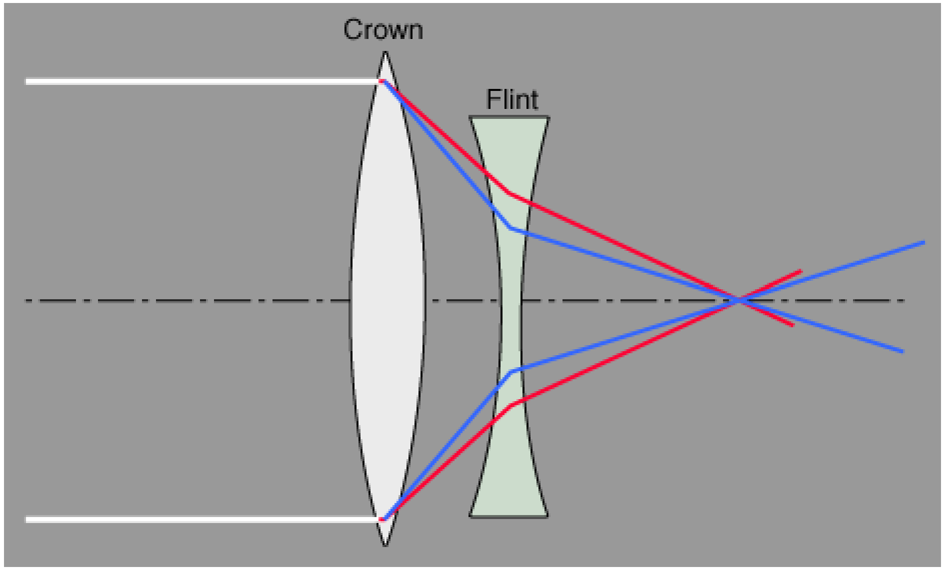
transverse chromatic aberration
(lateral)

Chromatic aberration is visible as color fringing around contrasty edges and occurs more frequently around the edges of the image frame in wide angle shots.

Chromatic Aberration

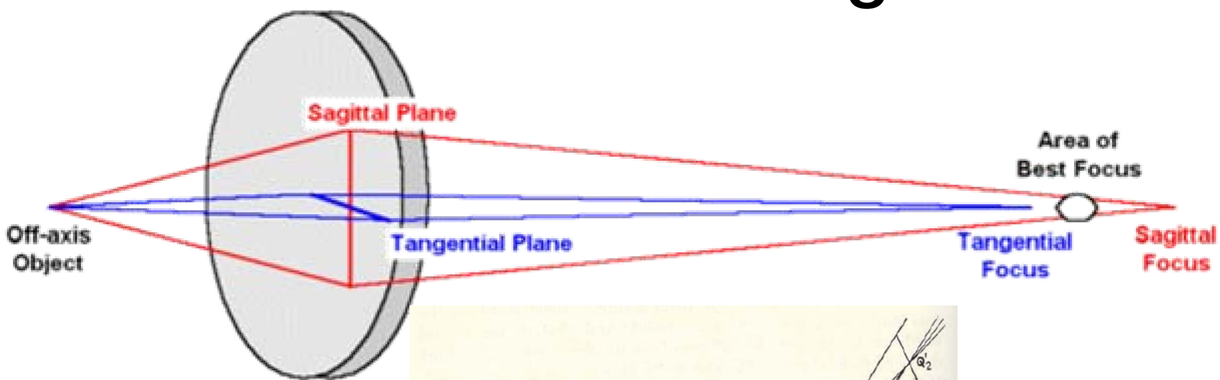
Doublet for Chromatic Aberration

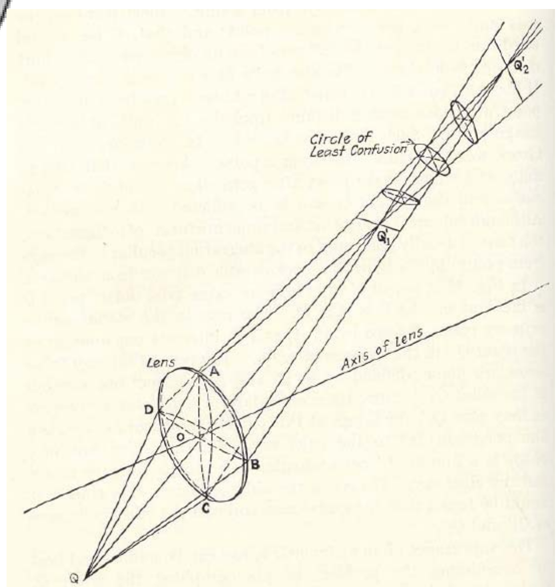
The use of a strong positive lens made from a low dispersion glass like crown glass coupled with a weaker high dispersion glass like flint glass can correct the chromatic aberration for two colors, e.g., red and blue.

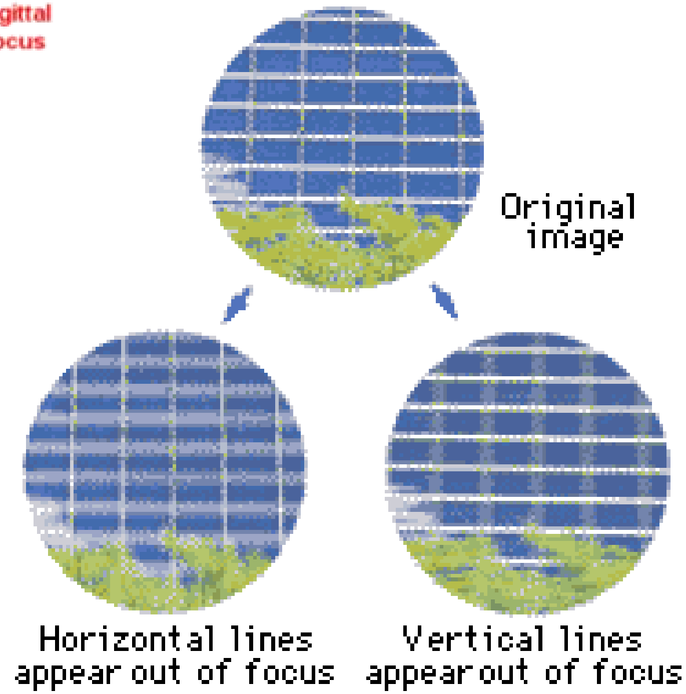


Astigmatism(像散)

Different focal length for inclined rays

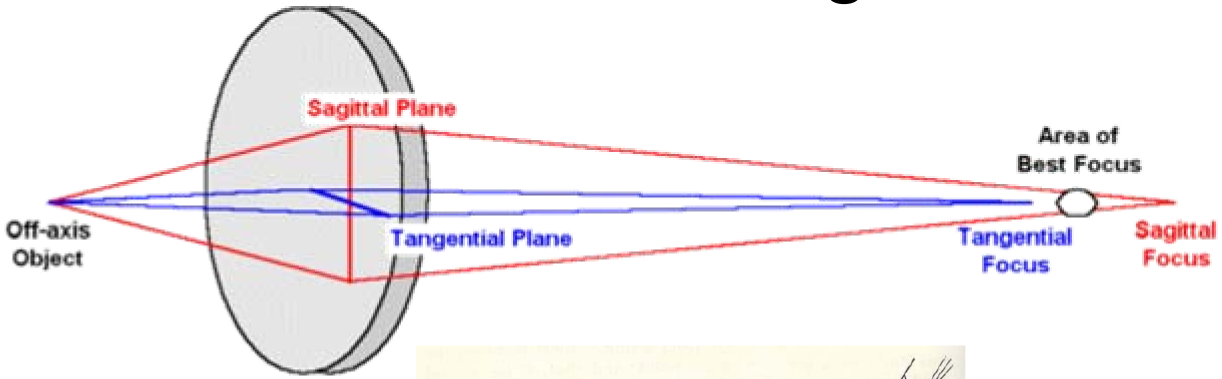


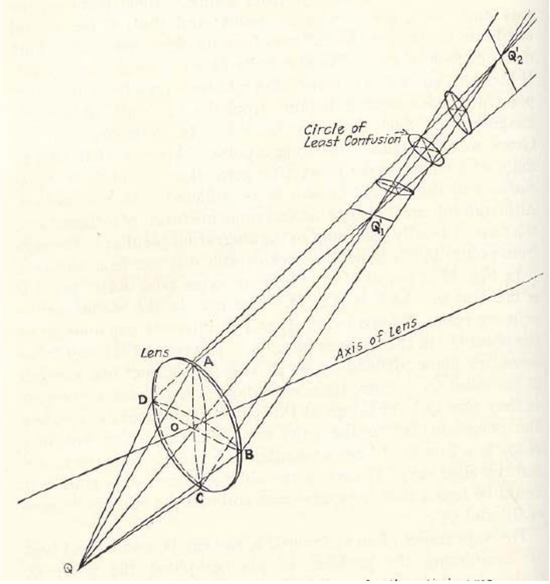


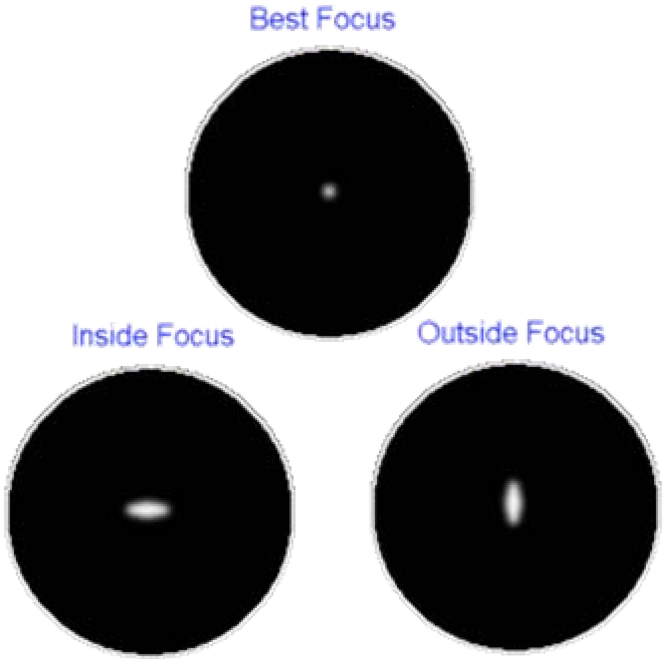


Astigmatism

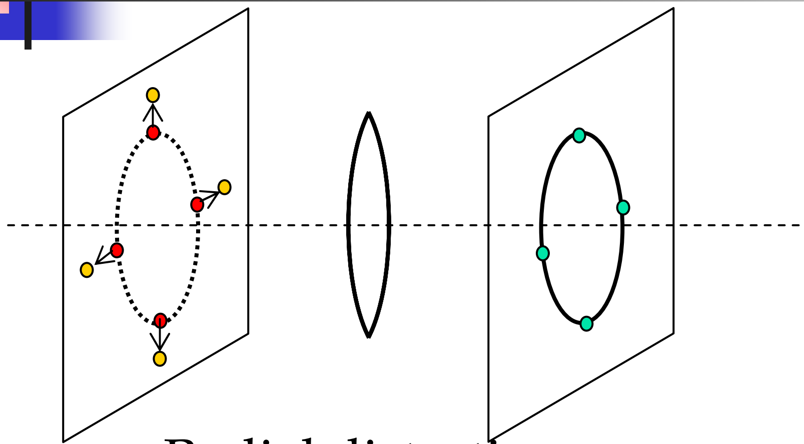
Different focal length for inclined rays



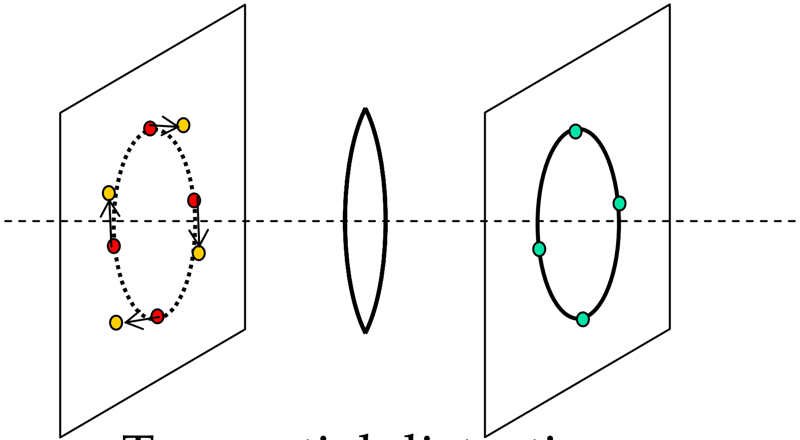




Geometric Distortion



Radial distortion



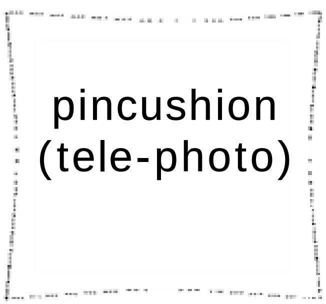
Tangential distortion

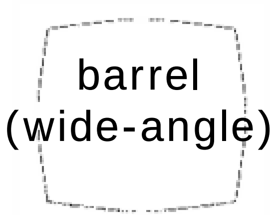
Photo by Helmut Dersch

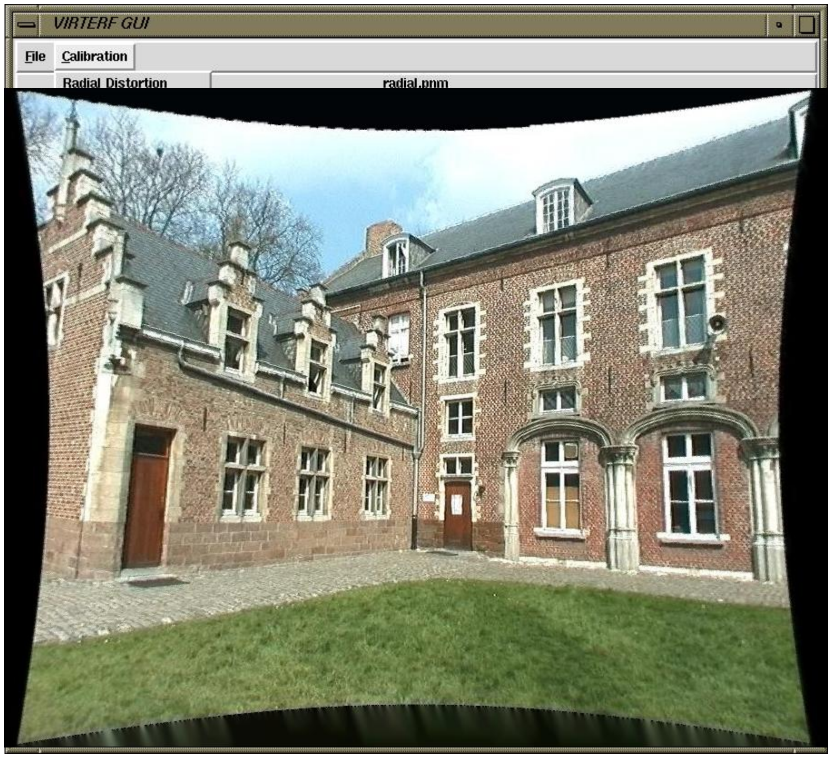
Both due to lens imperfection. Rectify with geometric camera calibration

Distortion

magnification/focal length different for different angles of inclination





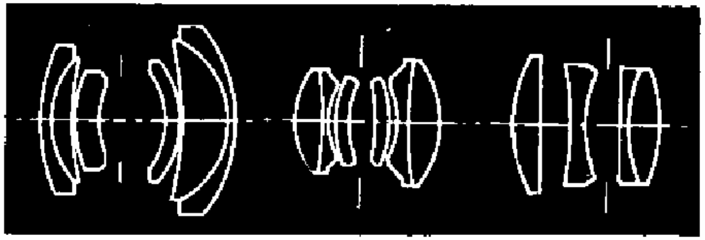


Can be corrected! (if parameters are know)

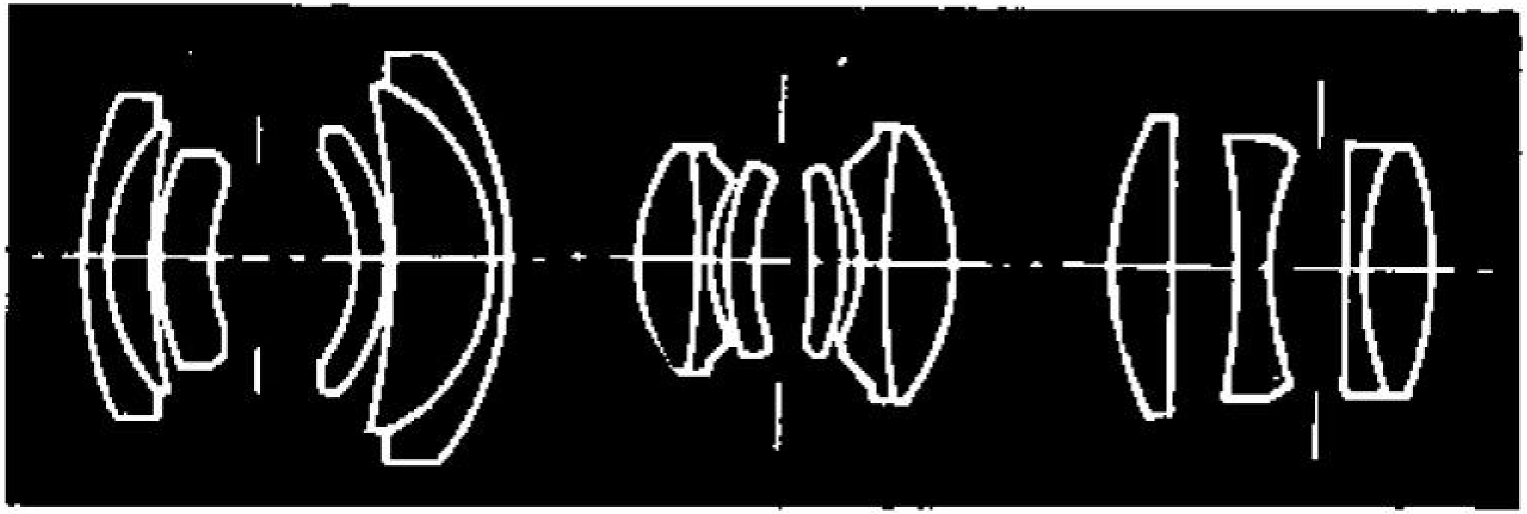
Geometrical Aberrations

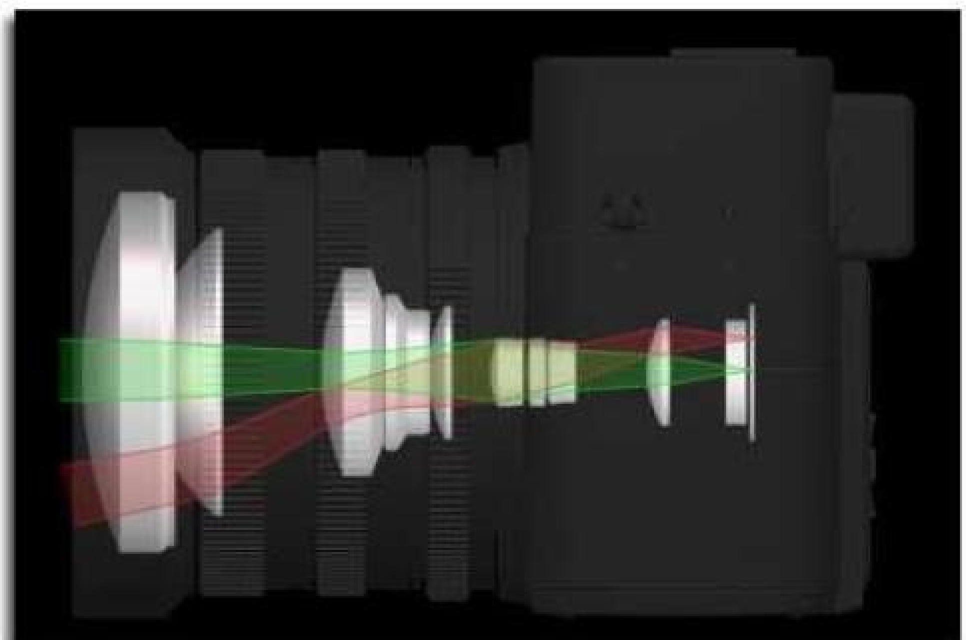
- spherical aberration
- astigmatism
- distortion
- coma

aberrations are reduced by combining lenses



Lens systems





- A good camera lens may contain 15 elements and cost a thousand dollars
- The best modern lenses may contain aspherical elements

Interaction of light with matter

- Absorption
- Scattering
- Refraction
- Reflection
- Other effects:
  - Diffraction: deviation of straight propagation in the presence of obstacles
  - Fluorescence: absorbtion of light of a given wavelength by a fluorescent molecule causes reemission at another wavelength

CCD vs. CMOS

- Mature technology
- Specific technology
- High production cost
- High power consumption
- Higher fill rate
- Blooming
- Sequential readout

- Recent technology
- Standard IC technology
- Cheap
- Low power
- Less sensitive
- Per pixel amplification
- Random pixel access

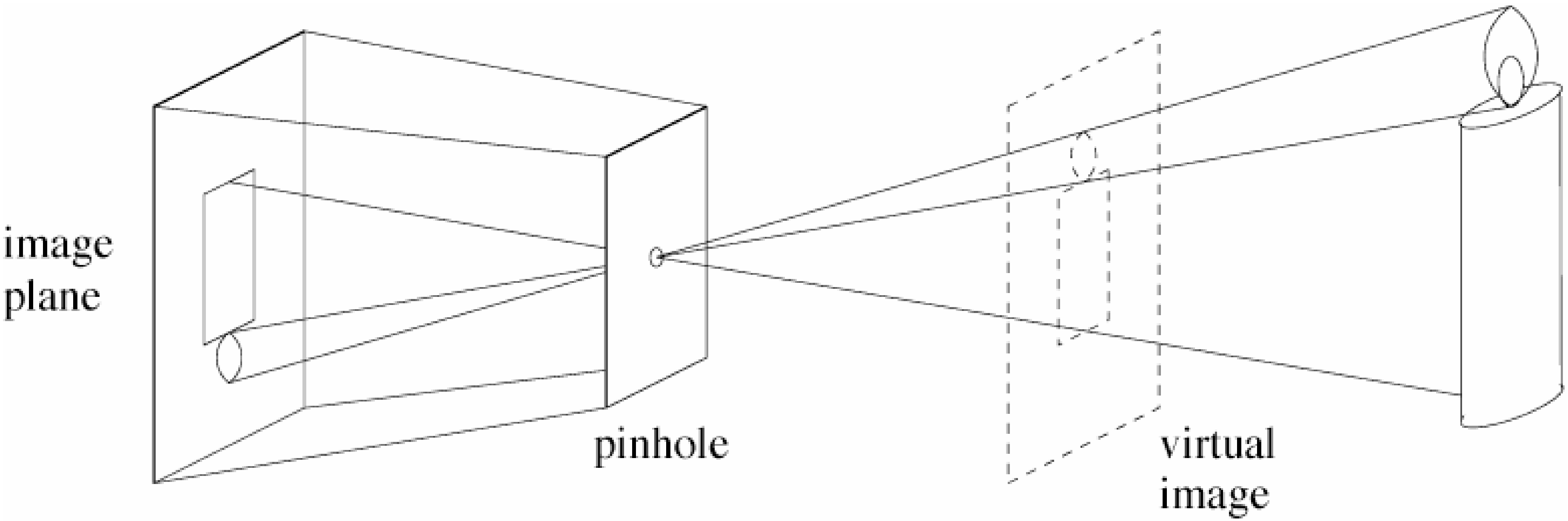
Camera Geometry (A BREAK)

Topics

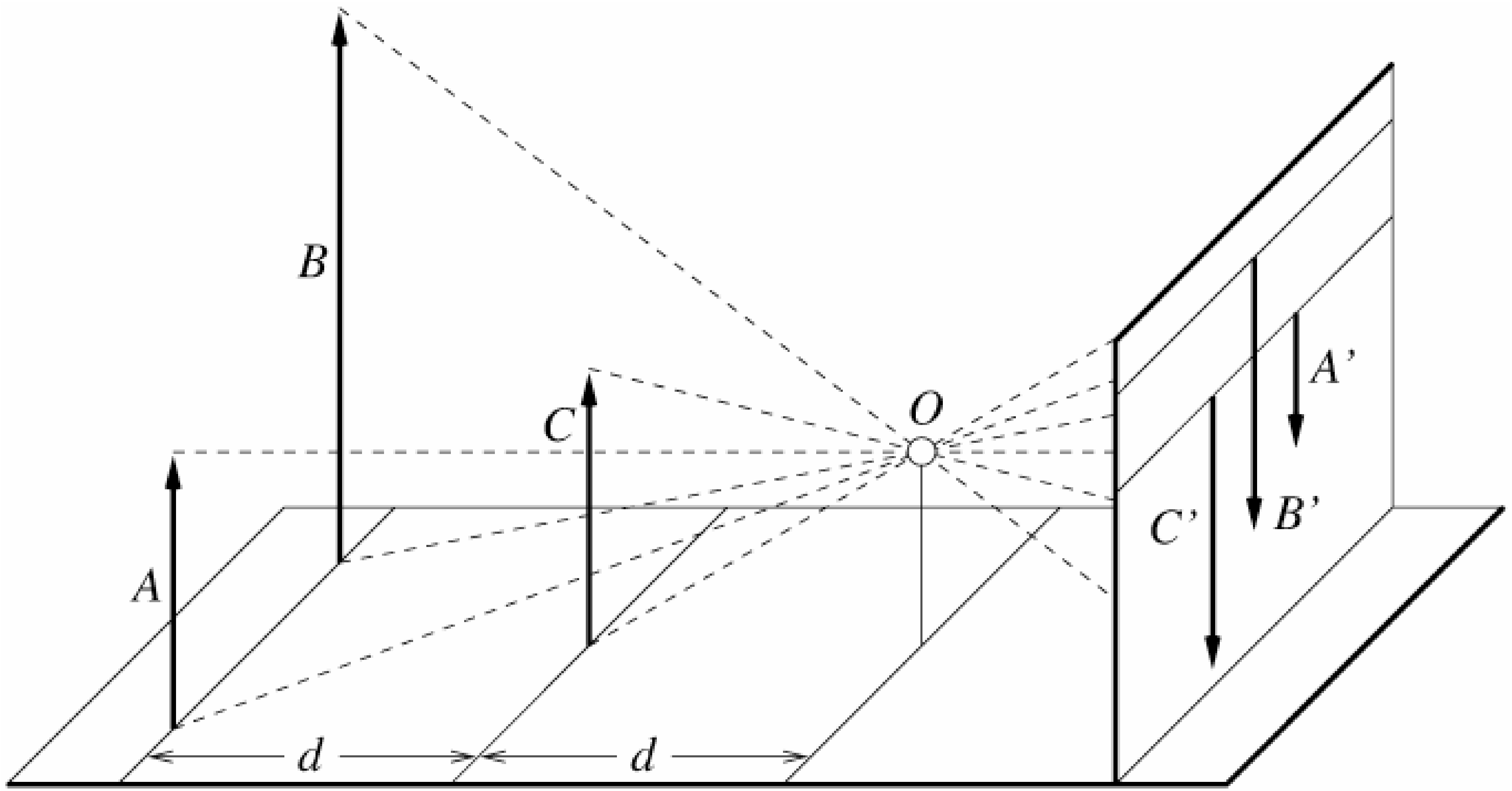
- Pinhole Camera
- Orthographic Projection
- Perspective Camera Model
- Weak-Perspective Camera Model

Pinhole cameras

- Abstract camera model - box with a small hole in it
- Pinhole cameras work in practice

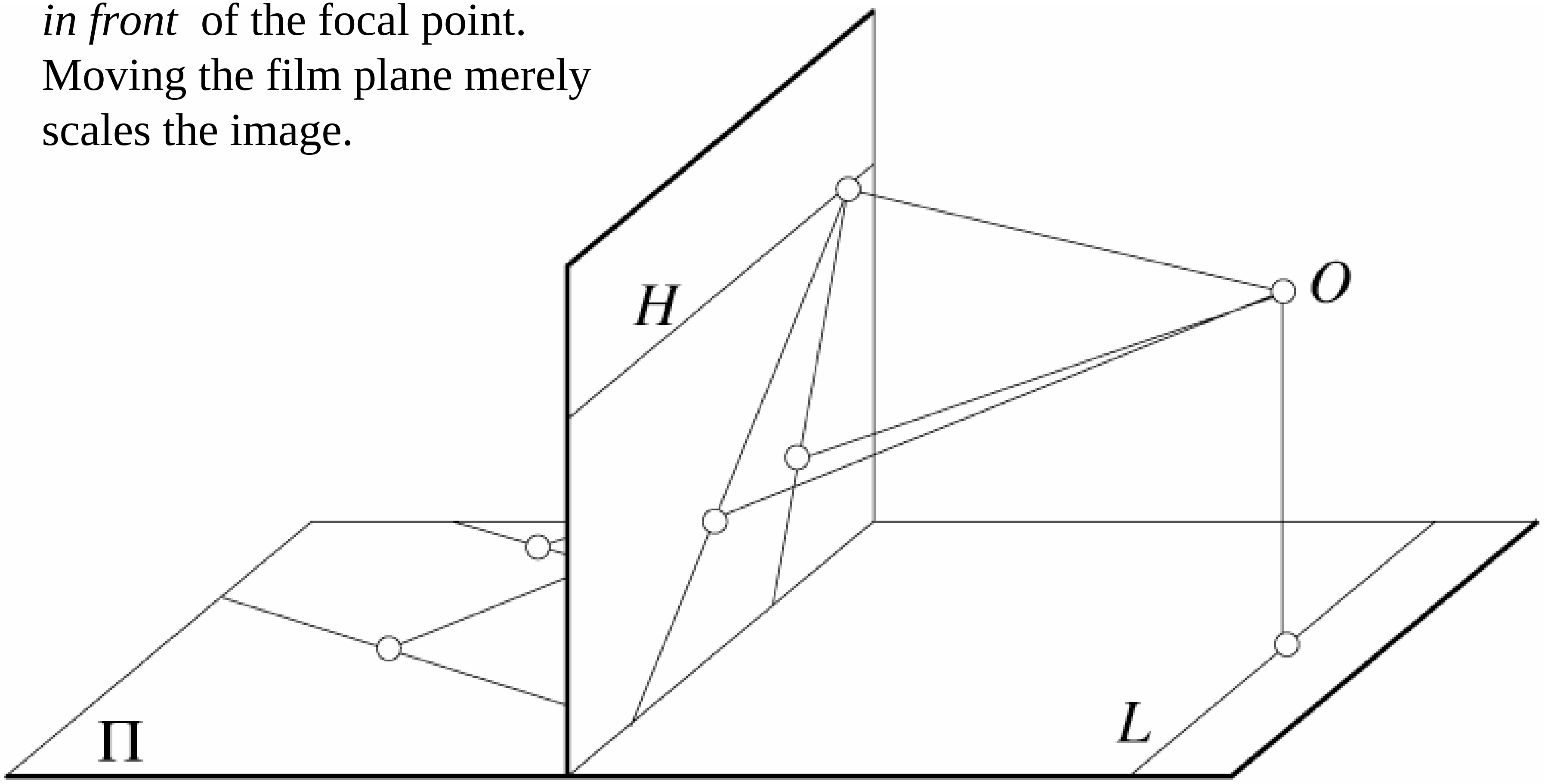


Distant objects are smaller



Parallel lines meet

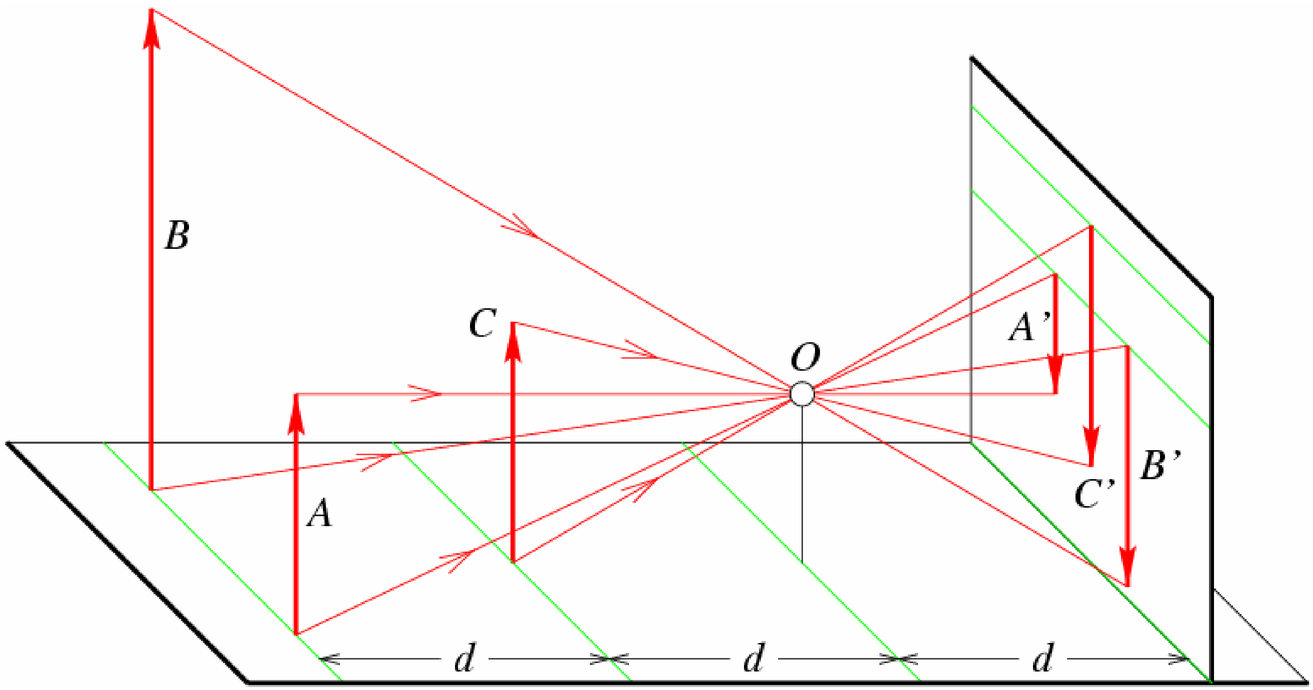
Common to draw film plane
in front of the focal point.
Moving the film plane merely
scales the image.



Vanishing points

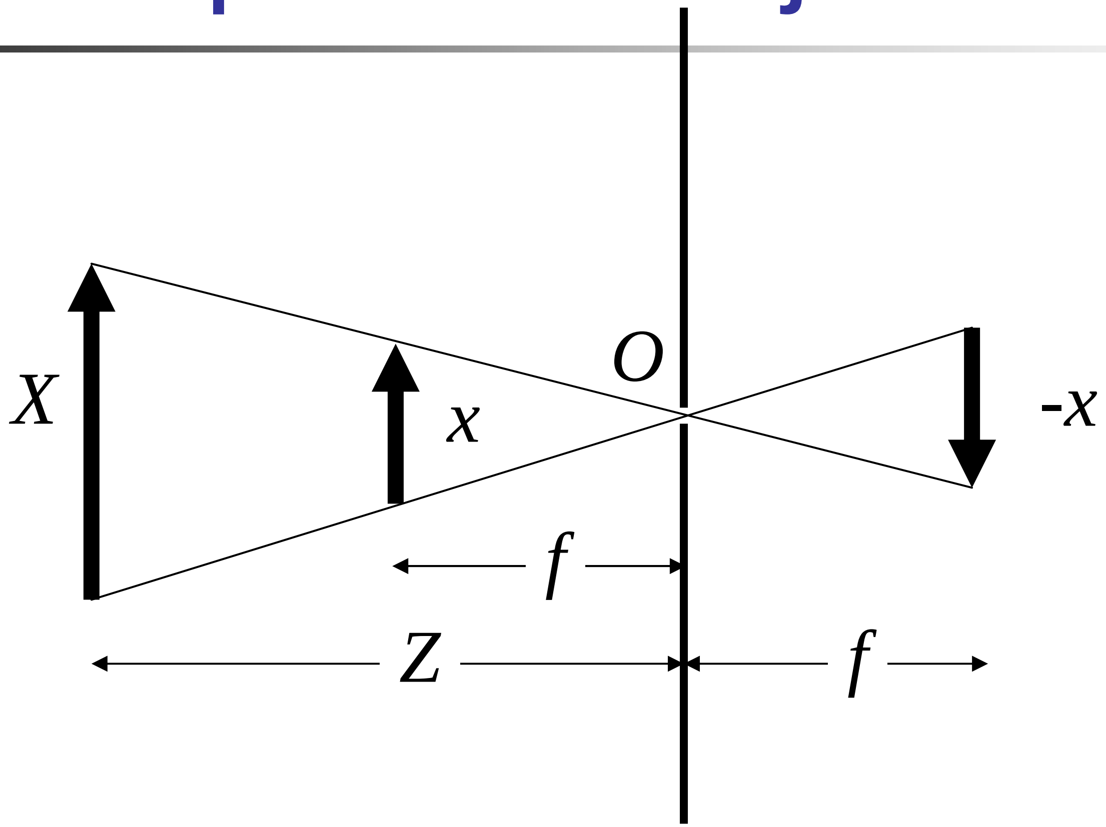
- Each set of parallel lines meets at a different point
  - The *vanishing point* for this direction
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon (or vanishing line)* for that plane

Perspective Projection



A “similar triangle’s” approach to vision.

Perspective Projection



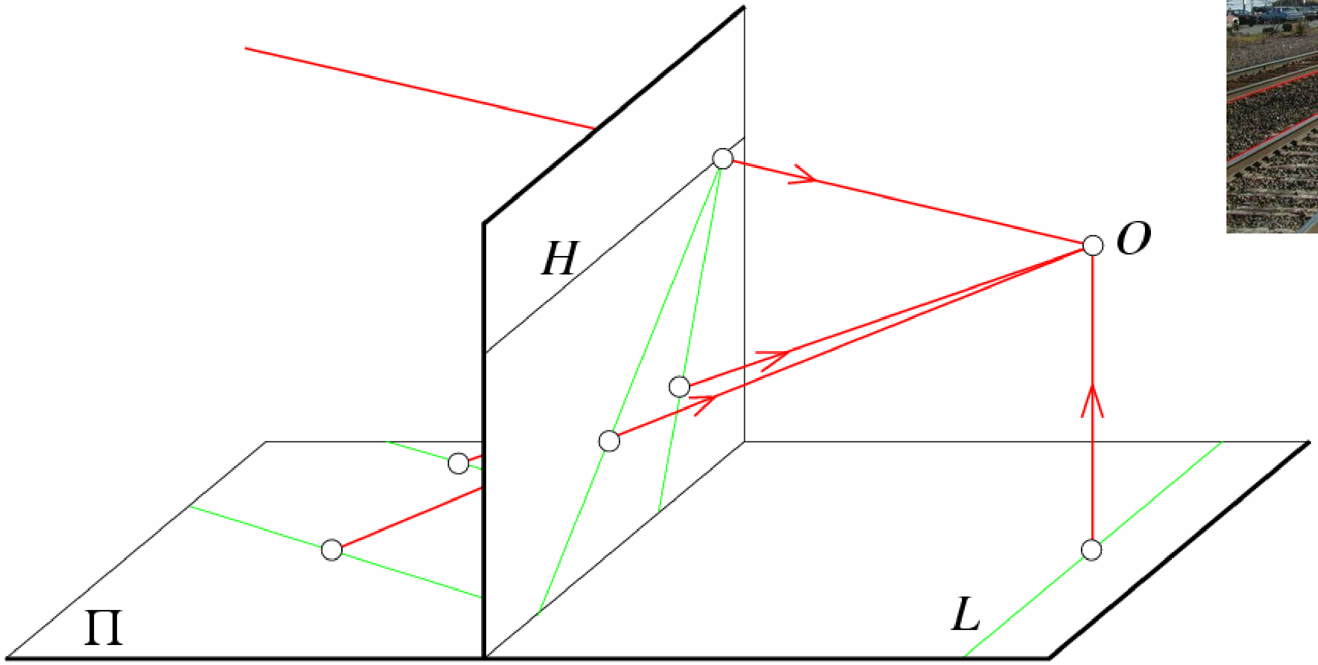
$$x = -X \frac{f}{Z}$$

Properties of Projection

- Points project to points
- Lines project to lines
- Planes project to the whole image or a half image
- Angles are not preserved
- Degenerate cases
  - Line through focal point projects to a point.
  - Plane through focal point projects to line

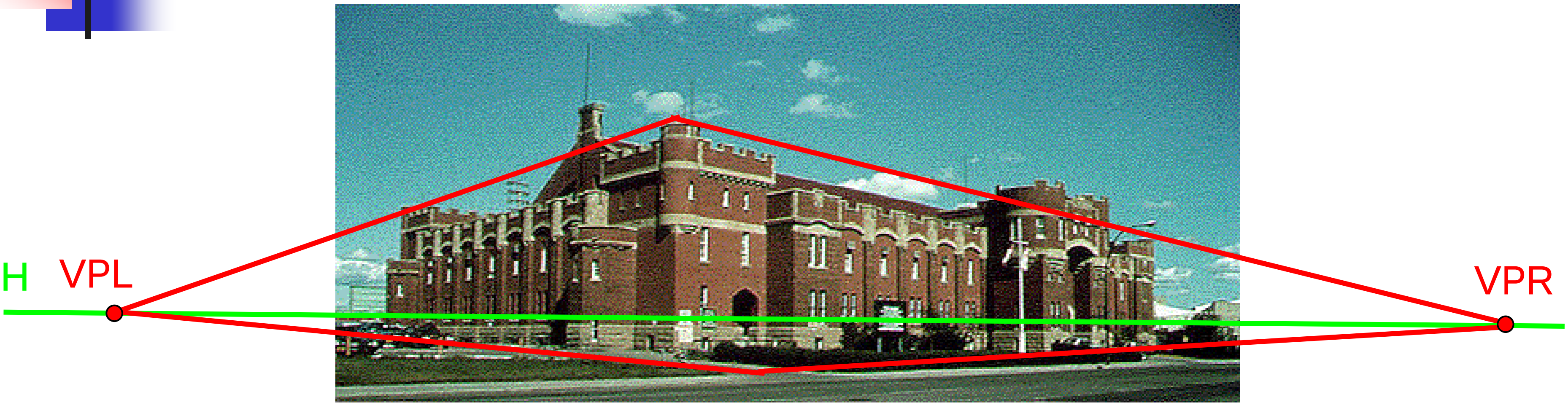
Consequences: Parallel lines meet

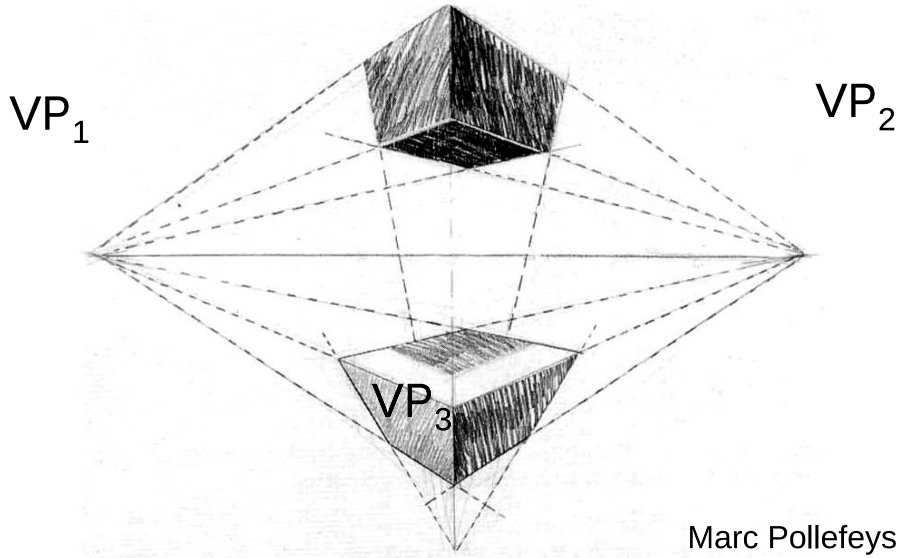
- There exist vanishing points



The Effect of Perspective

Vanishing points





Different directions correspond to different vanishing points

Marc Pollefeys

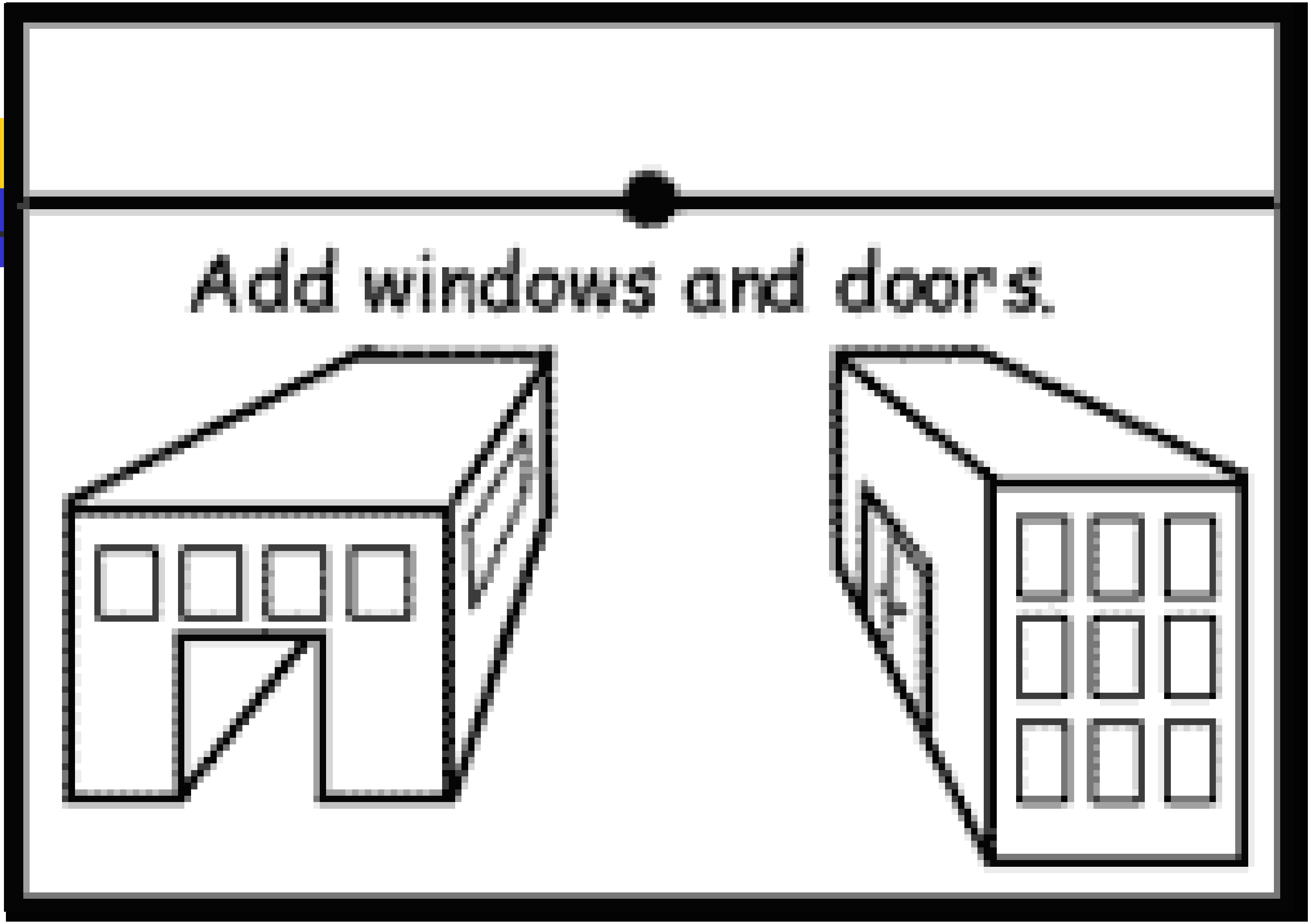
Same size things get smaller, we hardly notice…

Parallel lines meet at a point…

* A Cartoon Epistemology: http://cns-alumni.bu.edu/~slehar/cartoonepist/cartoonepist.html

Try it yourself

Take out paper and pencil and rubber



http://www.sanford-artedventures.com/create/tech_1pt_perspective.html

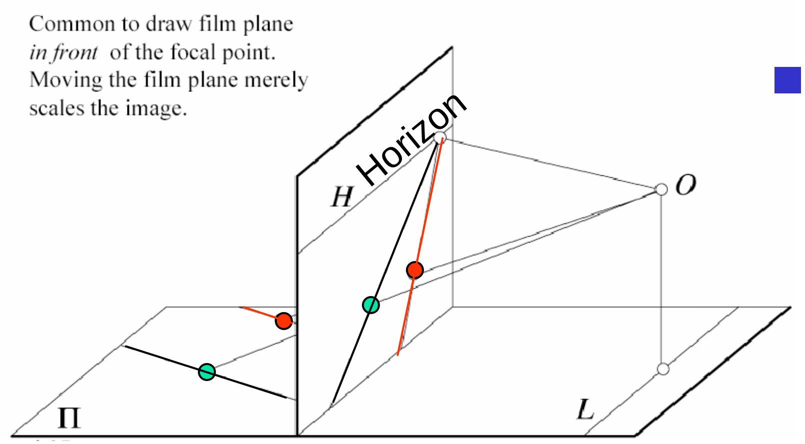
Assignment/Project

- Any work related to image and vision computing is acceptable
- Presentation at the end of the semester
  - Pre-submission of demos, codes, and documents
  - PPT and DEMO at presentation
  - Each student has around 30 minutes
- Good work win exemption of final exam

Assignment/Project presentation

- June 3rd, Tuesday, 16:50-18:40
- Presentation: powerpoint slides, 30 minutes
- Presubmission (By June 1st, 23:59):
  - Codes (concise) with comments (Detailed)
    - Distinct your work from open sources (if any)
  - Result demos (Mandatory)
  - Documents (Emphasis on your own work)
- The sooner, the better.

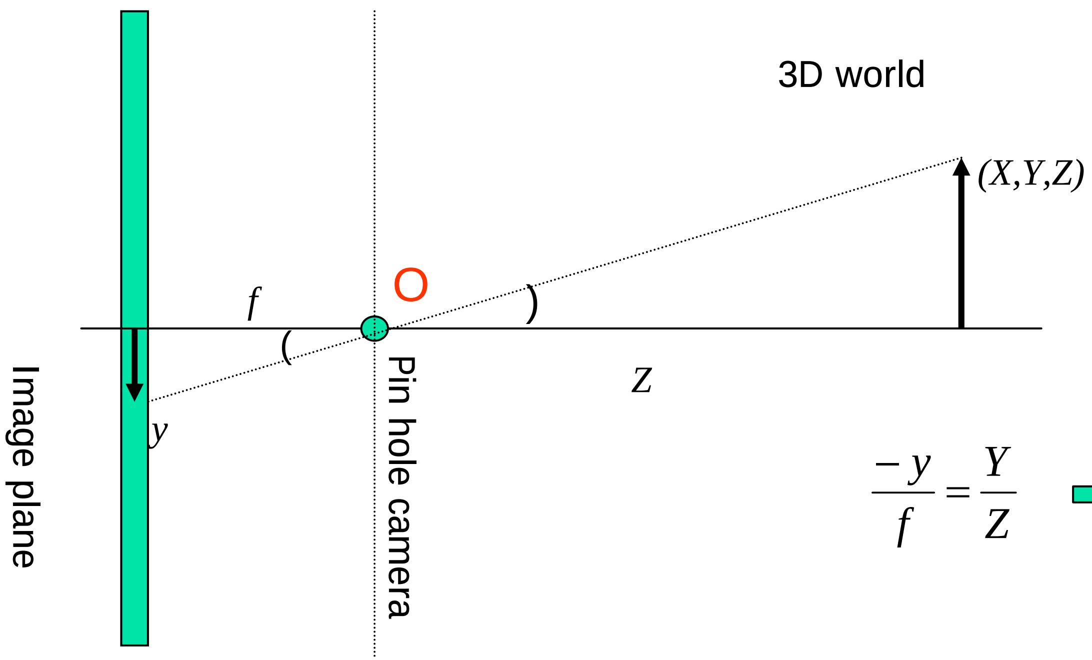
Perspective Projection



- Objects farther appear smaller
- Points go to Points
- Lines go to Lines
- Polygons go to Polygons
- Parallel lines meet

Appendix

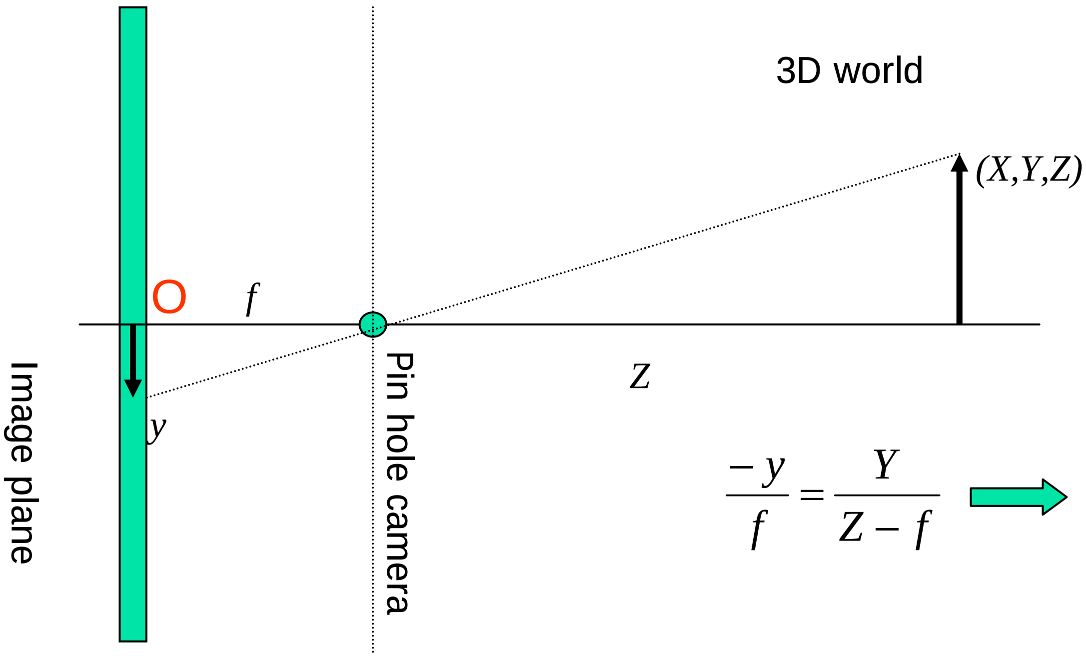
Perspective Projection (Origin at lens center)



$$\frac{-y}{f} = \frac{Y}{Z} \Rightarrow y = -f\,\frac{Y}{Z}$$

$$x = -f\,\frac{X}{Z}$$

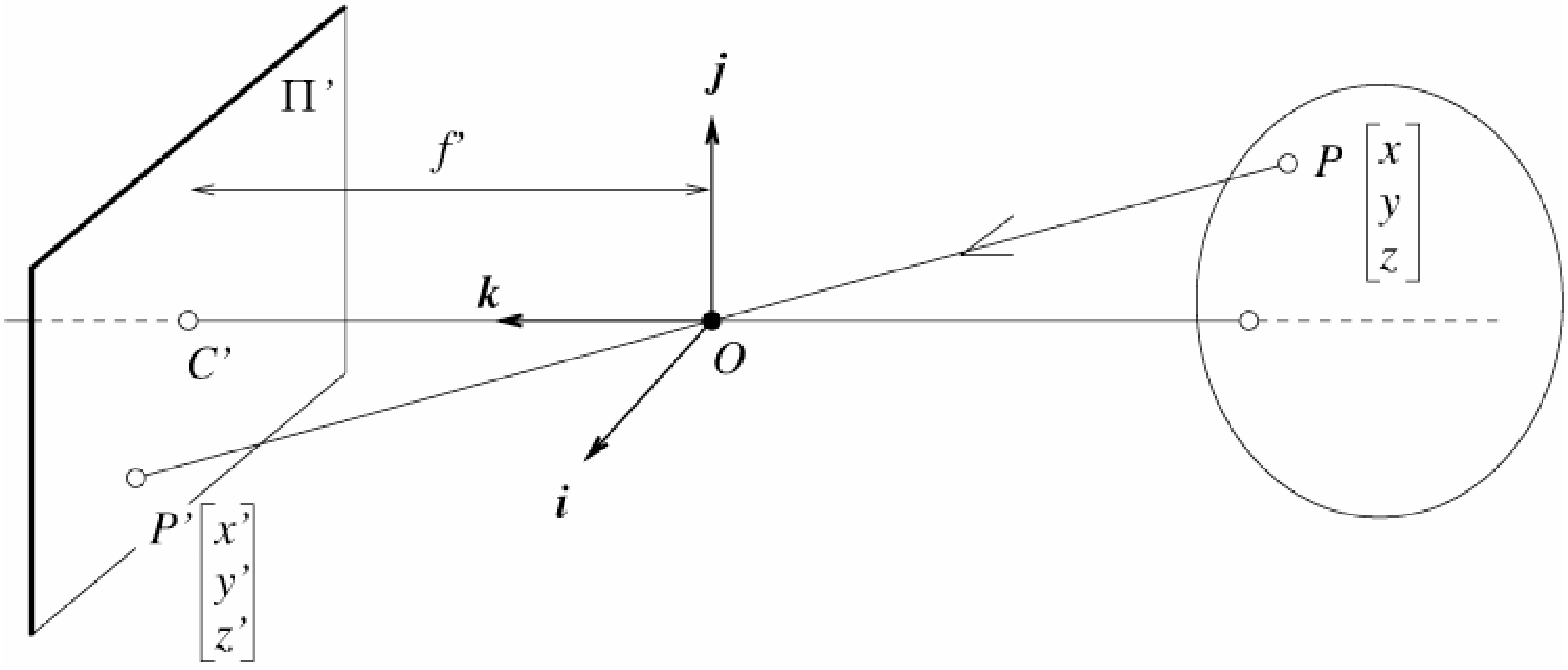
Perspective Projection (Origin at image center)



$$\frac{-y}{f} = \frac{Y}{Z-f} \Rightarrow y = -f\frac{Y}{Z-f} \Rightarrow y = \frac{fY}{f-Z}$$

$$x = \frac{fX}{f-Z}$$

The equation of projection



(Forsyth & Ponce)

The equation of projection

- Cartesian coordinates:
  - We have, by similar triangles, that

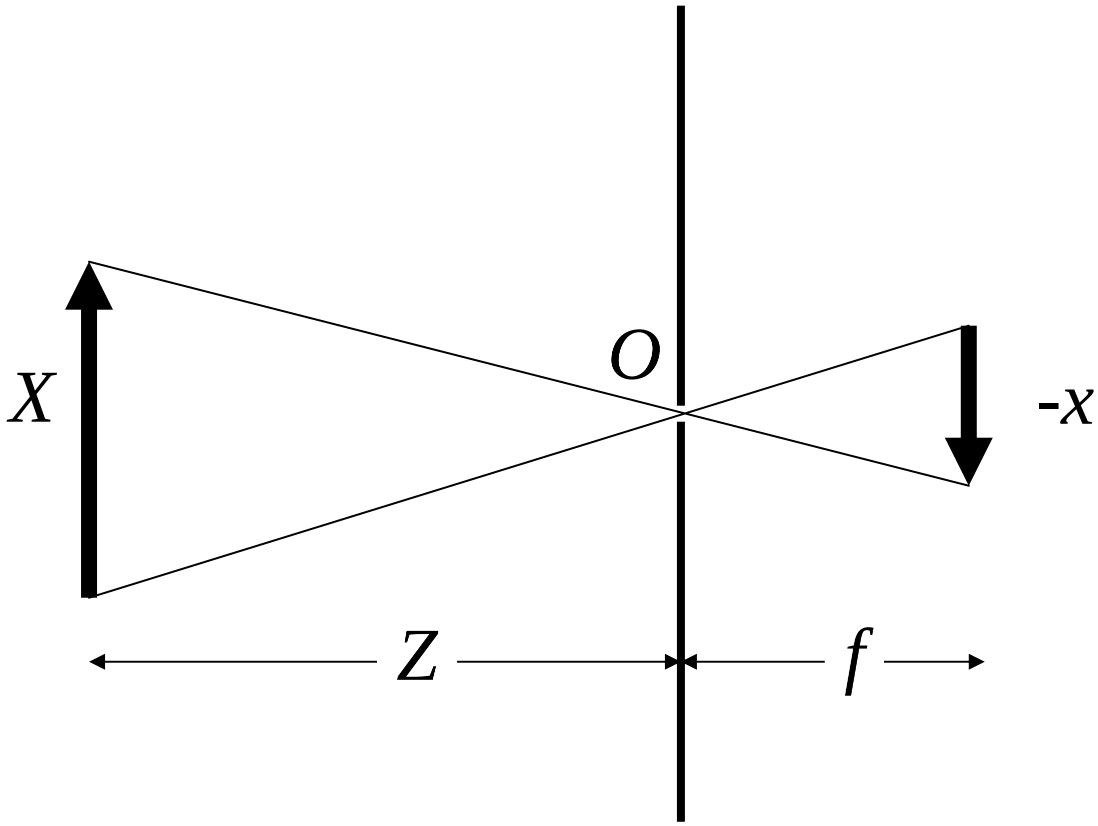
$$x' = f'\frac{x}{z}, y' = f'\frac{y}{z}$$

$$(x, y, z) \rightarrow (f'\frac{x}{z}, f'\frac{y}{z}, f')$$

  - Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f'\frac{x}{z}, f'\frac{y}{z})$$

Perspective Projection



$$x = X\frac{f}{Z}$$

Homogenous coordinates

- Add an extra coordinate and use an equivalence relation
- for 2D
  - equivalence relation k*(X,Y,Z) is the same as (X,Y,Z)
- for 3D
  - equivalence relation k*(X,Y,Z,T) is the same as (X,Y,Z,T)
- Basic notion
  - Possible to represent points “at infinity”
    - Where parallel lines intersect
    - Where parallel planes intersect
  - Possible to write the action of a perspective camera as a matrix

Homogenous Coordinates

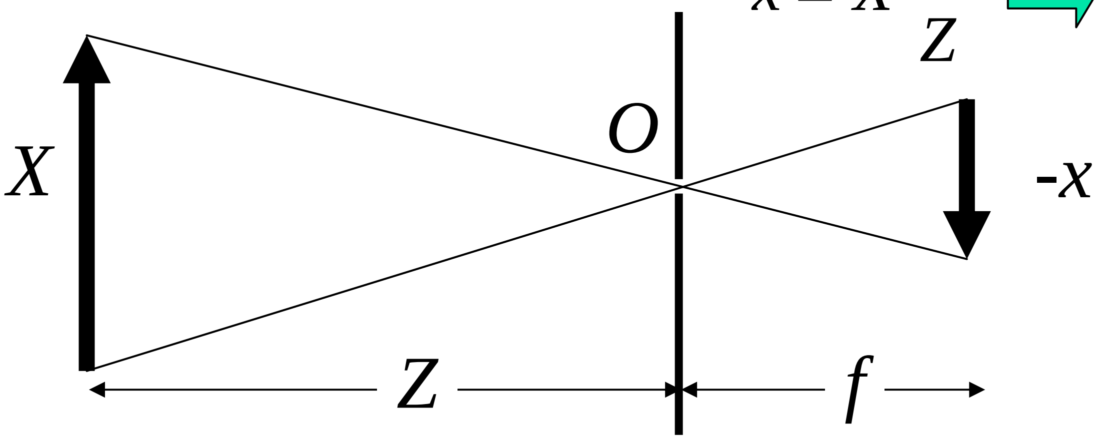
$$\overbrace{(X,Y,Z)}^{\text{cartesian world coordinates}} \Rightarrow \overbrace{(kX,kY,kZ,k)}^{\text{homogenous world corrdinates}}$$

$$\overbrace{(C_1,C_2,C_3,C_4)}^{\text{homogenous world corrdinates}} \Rightarrow \overbrace{(\frac{C_1}{C_4},\frac{C_2}{C_4},\frac{C_3}{C_4})}^{\text{cartesian world coordinates}}$$

Homogeneous camera matrix

- Turn previous expression into HC's
  - HC's for 3D point are (X,Y,Z,T)
  - HC's for point in image are (U,V,W)

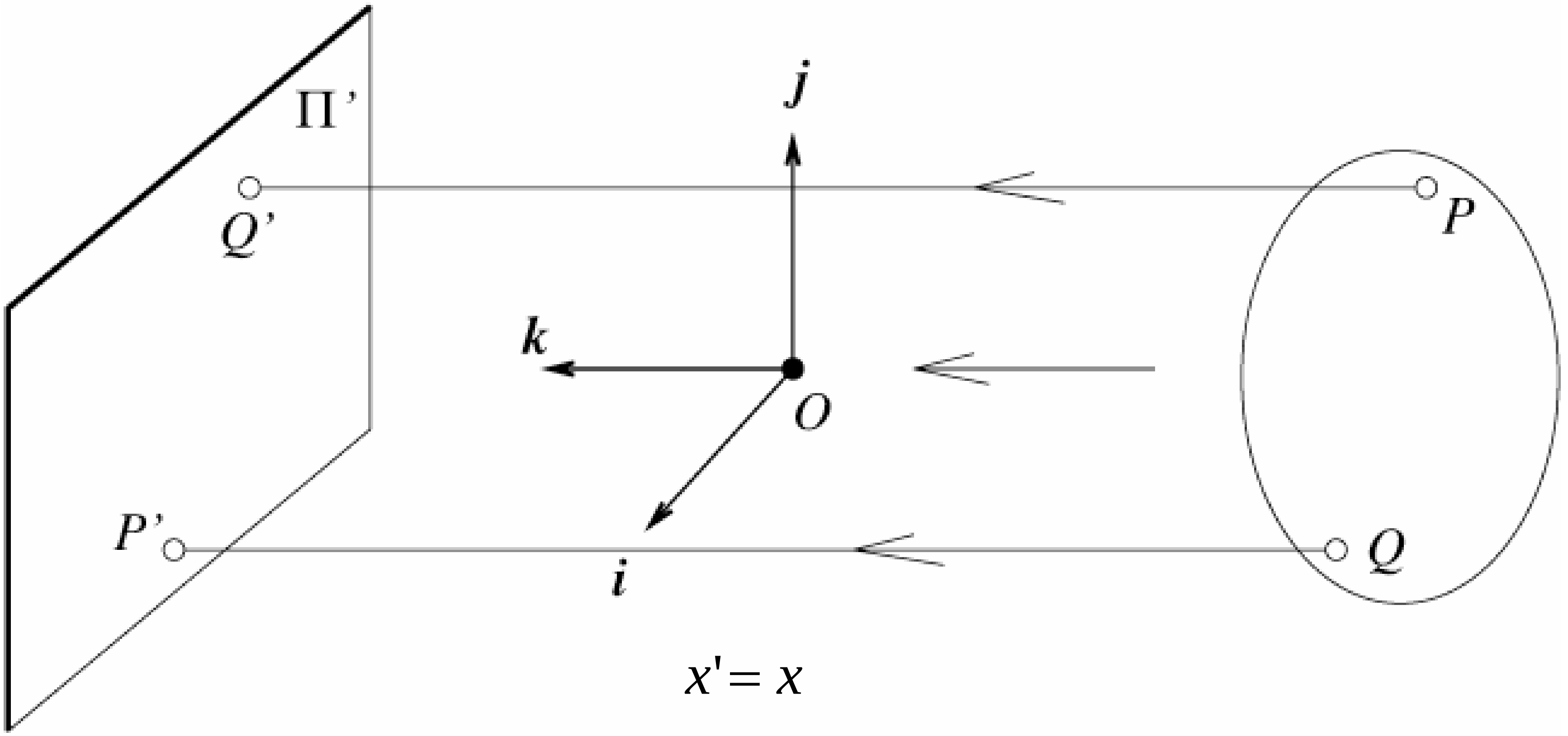
$$x = X\frac{f}{Z}$$



$$x = \frac{U}{W} = X\frac{f}{Z} \qquad y = \frac{V}{W} = Y\frac{f}{Z}$$

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Orthographic projection



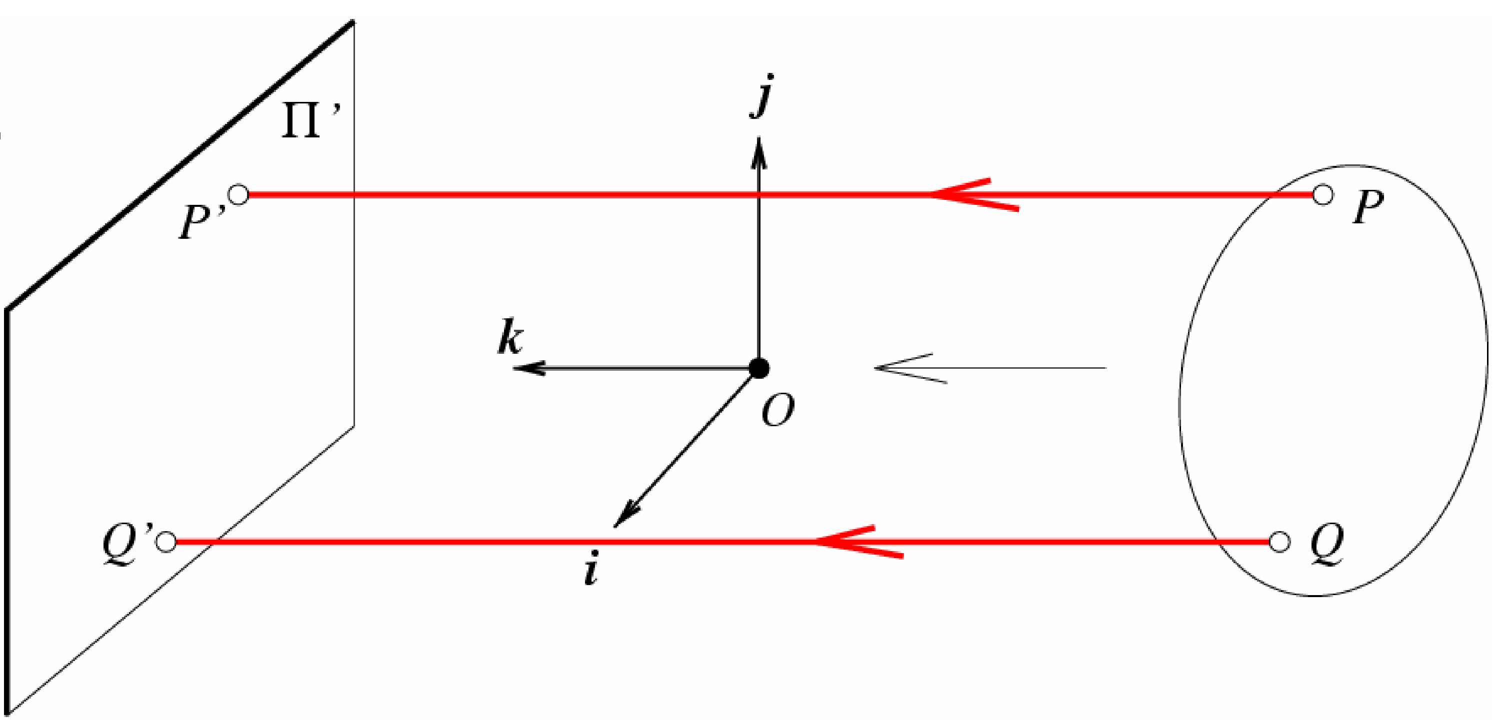
$$x' = x$$

$$y' = y$$

The projection matrix for orthographic projection

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Generalization of Orthographic Projection

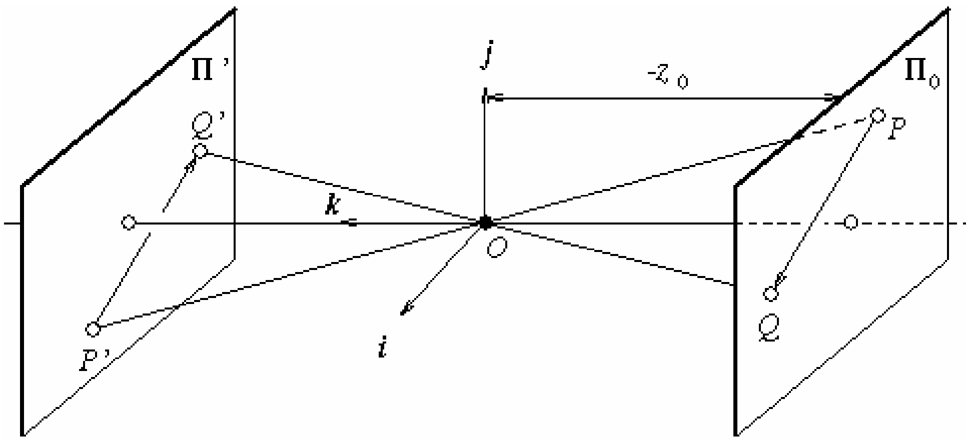


$$\boxed{\begin{cases} X = x \\ Y = y \end{cases}}$$

When the camera is at a (roughly constant) distance from the scene, take *m*=1.

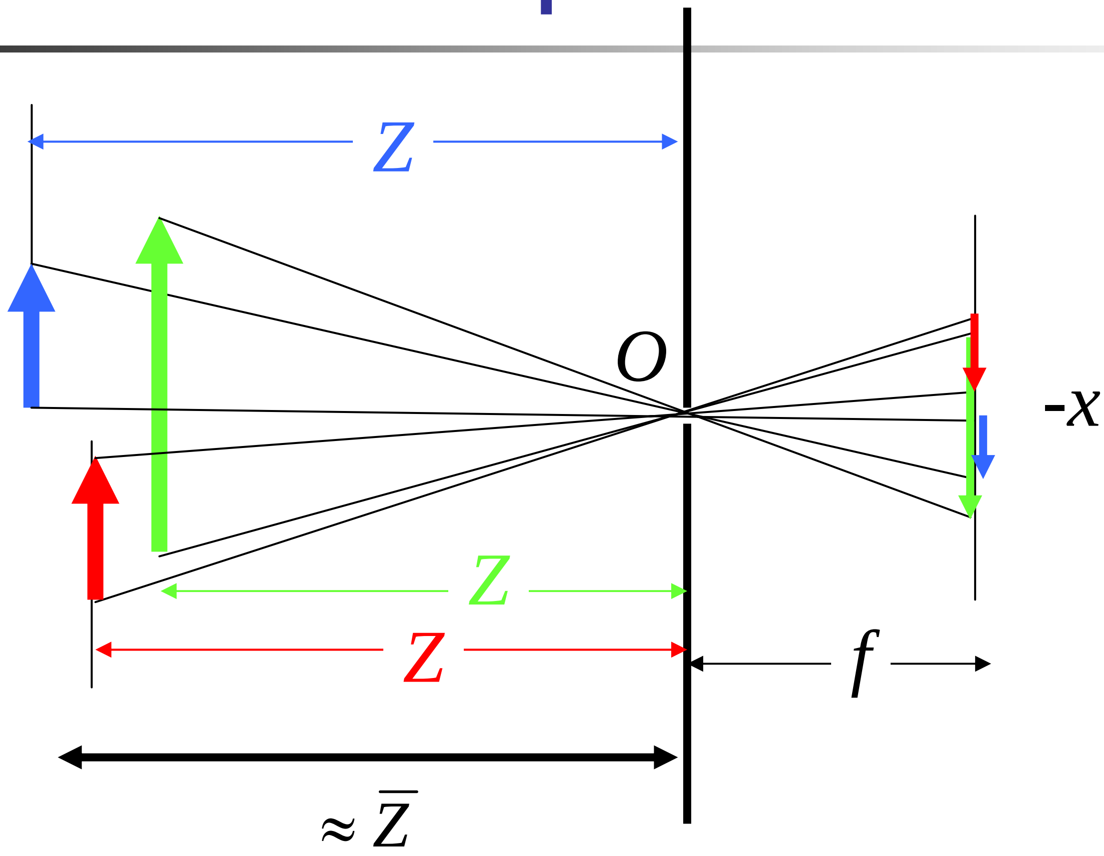
Weak perspective (scaled orthographic projection)

- Issue
  - perspective effects, but not over the scale of individual objects
  - collect points into a group at about the same depth, then divide each point by the depth of its group



(Forsyth & Ponce)

Weak Perspective Projection



$$x \approx X \frac{f}{\bar{Z}}$$

$$= const \cdot X$$

The Equation of Weak Perspective

$$(x, y, z) \rightarrow s(x, y)$$

- *s* is constant for all points.
- Parallel lines no longer converge, they remain parallel.

Homogeneous representation

Orthographic:

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Weak Perspective

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Pictorial Comparison

Weak perspective

Perspective

Marc Pollefeys

Summary: Perspective Laws

1. Perspective $x = X \frac{f}{Z} \qquad y = Y \frac{f}{Z}$

2. Weak perspective $x = const\ X, y = const\ Y$

3. Orthographic $x = X \qquad y = Y$

$$x, y = \text{image coordinates}$$
$$X, Y, Z = \text{world coordinates}$$
$$Z = \text{depth}$$
$$f = \text{focal length of the camera}$$

Pros and Cons of These Models

- Weak perspective much simpler math.
  - Accurate when object is small and distant.
  - Most useful for recognition.
- Pinhole perspective much more accurate for scenes.
  - Used in structure from motion.
- When accuracy really matters, must model real cameras.

Camera Parameters

- Image coordinates (x_{image}, y_{image})
- Image center (o_x, o_y)
- Camera coordinates (x_{camera}, y_{camera})
- Real world coordinates (X, Y, Z)
- Focal length f
- Effective size of pixel (k_x, k_y)

Camera Parameters

$$x_{image} = k_x x_{camera} + o_x$$
$$y_{image} = k_y y_{camera} + o_y$$

$$\begin{bmatrix} x_{image} \\ y_{image} \\ 1 \end{bmatrix} = \begin{bmatrix} k_x & 0 & o_x \\ 0 & k_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{camera} \\ y_{camera} \\ 1 \end{bmatrix}$$

$$x = X\frac{f}{Z} \qquad y = Y\frac{f}{Z}$$

$$\begin{bmatrix} U_{image} \\ V_{image} \\ S \end{bmatrix} = \begin{bmatrix} k_x & 0 & o_x \\ 0 & k_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{camera} \\ V_{camera} \\ S \end{bmatrix} \qquad \begin{matrix} \boxed{-fX} \\ \boxed{-fY} \\ \boxed{Z} \end{matrix}$$

Camera Parameters

$$\begin{bmatrix} U_{image} \\ V_{image} \\ S \end{bmatrix} = \begin{bmatrix} k_x & 0 & o_x \\ 0 & k_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{camera} \\ V_{camera} \\ S \end{bmatrix}$$

$$\begin{bmatrix} U_{image} \\ V_{image} \\ S \end{bmatrix} = \begin{bmatrix} k_x & 0 & o_x \\ 0 & k_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera Parameters

$$\begin{bmatrix} U_{image} \\ V_{image} \\ S \end{bmatrix} = \begin{bmatrix} k_x & 0 & o_x \\ 0 & k_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -f & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} U_{image} \\ V_{image} \\ S \end{bmatrix} = \begin{bmatrix} -fk_x & 0 & o_x & 0 \\ 0 & -fk_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\boxed{f_x = fk_x}$$

$$\boxed{f_y = fk_y}$$

Intrinsic Camera Parameters

- f_x
- f_y
- o_x
- o_y
- Intrinsic parameters do not depend on camera position in real world.

Extrinsic Camera Parameters

- Defined by orientation of camera in real world
  - Translation (3x1 vector)
  - Rotation (3x3 matrix)

Translation

- (t_x, t_y, t_z) Translation vector

$$\begin{bmatrix} X_{camera} \\ Y_{camera} \\ Z_{camera} \end{bmatrix} = \begin{bmatrix} X_{world} \\ Y_{world} \\ Z_{world} \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \Rightarrow \begin{bmatrix} X_{camera} \\ Y_{camera} \\ Z_{camera} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation Matrix}} \begin{bmatrix} X_{world} \\ Y_{world} \\ Z_{world} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_{camera} \\ Y_{camera} \\ Z_{camera} \\ 1 \end{bmatrix} = T \begin{bmatrix} X_{world} \\ Y_{world} \\ Z_{world} \\ 1 \end{bmatrix}$$

Translation

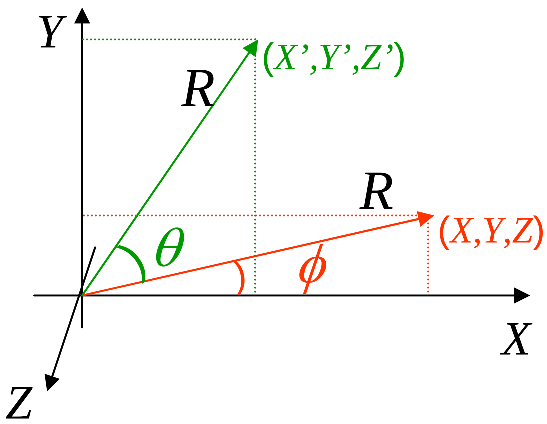
- Inverse translation

$$T=\begin{bmatrix}1&0&0&t_x\\0&1&0&t_y\\0&0&1&t_z\\0&0&0&1\end{bmatrix} \Rightarrow T^{-1}=\begin{bmatrix}1&0&0&-t_x\\0&1&0&-t_y\\0&0&1&-t_z\\0&0&0&1\end{bmatrix}$$

$$TT^{-1}=\begin{bmatrix}1&0&0&t_x\\0&1&0&t_y\\0&0&1&t_z\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-t_x\\0&1&0&-t_y\\0&0&1&-t_z\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$$

Rotation

- Around Z-axis



$$X = R\cos\phi$$

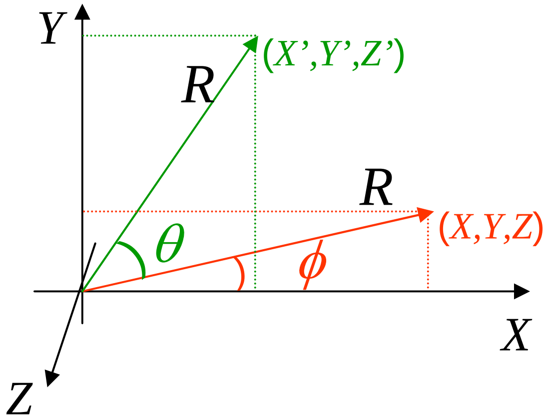
$$Y = R\sin\phi$$

$$X' = R\cos(\phi+\theta) = \overbrace{R\cos\phi}^{X}\cos\theta - \overbrace{R\sin\phi}^{Y}\sin\theta$$

$$Y' = R\sin(\phi+\theta) = \overbrace{R\cos\phi}^{X}\sin\theta + \overbrace{R\sin\phi}^{Y}\cos\theta$$

Rotation

- Around Z-axis



$$X' = X\cos\theta - Y\sin\theta$$

$$Y' = X\sin\theta + Y\cos\theta$$

$$\begin{bmatrix} X' \\ Y' \\ Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Rotation

- Around X-axis

$$R^X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Around Y-axis

$$R^Y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

- Around Z-axis

$$R^Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- No rotation

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Inverse rotation

$$R^Z.\left(R^Z\right)^T = I$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotation matrices are orthonormal!!

$$R_i^T \cdot R_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Euler Angles

- Let γ, β, α be rotation angles around X, Y, Z axis respectively.

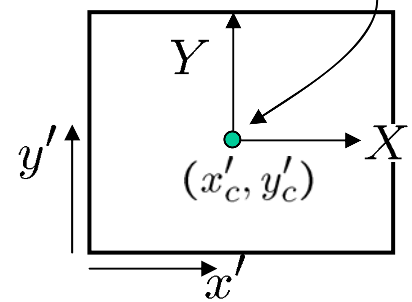
$$R = R_Z^{\alpha} R_Y^{\beta} R_X^{\gamma}$$

Camera parameters

A camera is described by several parameters

- Translation T of the optical center from the origin of world coords
- Rotation R of the image plane
- focal length f, principle point (x'_c, y'_c), pixel size (s_x, s_y)
- blue parameters are called "extrinsics," red are "intrinsics"

Projection equation



$$\mathbf{x} = \begin{bmatrix} sx \\ sy \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \mathbf{\Pi X}$$

- The projection matrix models the cumulative effect of all parameters
- Useful to decompose into a series of operations

identity matrix

$$\mathbf{\Pi} = \underbrace{\begin{bmatrix} -fs_x & 0 & x'_c \\ 0 & -fs_y & y'_c \\ 0 & 0 & 1 \end{bmatrix}}_{\text{intrinsics}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{projection}} \underbrace{\begin{bmatrix} \mathbf{R}_{3x3} & \mathbf{0}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \mathbf{I}_{3x3} & \mathbf{T}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix}}_{\text{translation}}$$

- The definitions of these parameters are **not** completely standardized
  - especially intrinsics—varies from one book to another