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Vertex PI indices of four sums of graphs☆

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ABSTRACT

Suppose that e is an edge of a graph G. Denote by $m_e(G)$ the number of vertices of G that are not equidistant from both ends of e. Then the vertex PI index of G is defined as the summation of $m_e(G)$ over all edges e of G. In this paper we give the explicit expressions for the vertex PI indices of four sums of two graphs in terms of other indices of two individual graphs, which correct the main results in a paper published in Ars Combin. 98 (2011).

1. Introduction

A topological index is a real number related to a molecular graph, which does not depend on the labeling or the pictorial representation of a graph. Several indices have been defined and have found applications as means for modeling chemical, pharmaceutical and other properties of molecules. The Wiener index, introduced in 1947 by Wiener as the path number for the characterization of alkanes, was the first topological index to be used in chemistry [18–20]. The Szeged index, introduced by Gutman [5], is closely related to the Wiener index [6,10,11,14]. Since the Szeged index takes into account how the vertices are distributed, it is natural to introduce an index that takes into account the distribution of edges. The PI index is a Szeged-like index that takes into account the distribution of edges and a unique topological index related to parallelism of edges too. The vertex PI index was introduced by Khalifeh et al. in [13]. Its definition is similar to that of the PI index, in that it is additive, but now the distances of vertices from edges are considered. All indices mentioned above have many chemical applications [1,3,7,9,16] and correlate with the physico-chemical properties and biological activities of a large number of diverse and complex compounds [8,12].

Wiener indices, and hyper-Wiener indices and reverse Wiener indices for four new sums of two graphs were computed in [4,17], respectively. Vertex PI indices of four sums of two graphs have been computed in [15], but the main results in [15] are wrong. In this paper we deal with the errors in [15] and give the correct expressions for their vertex PI indices in terms of other indices of two individual graphs.

2. Preliminaries

We first recall some operations on graphs in [2] (see Fig. 1).

Suppose that $G = (V, E)$ is a connected graph, and refer to each vertex of V as a *black vertex*. Then we denote by $S(G)$ the graph obtained from G by inserting an additional vertex which is referred to as the *white vertex* in each edge of G. Two black vertices in $S(G)$ are *related* if they are adjacent in G; and two white vertices in $S(G)$ are *related* if their corresponding edges in G are adjacent. Denote by $R(G)$ and $Q(G)$ the graphs obtained from $S(G)$ by joining every pair of related black vertices

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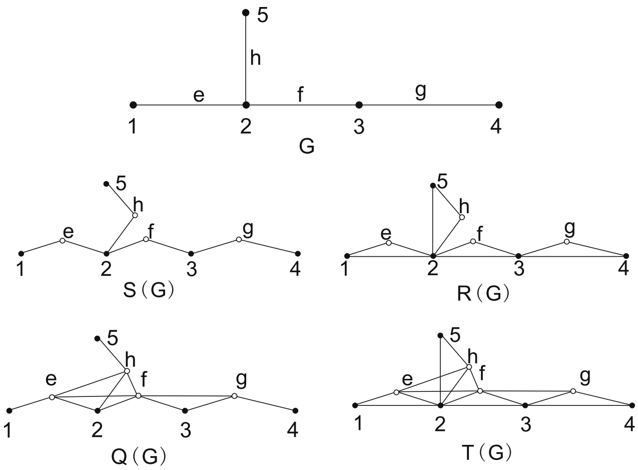

Fig. 1. A graph G and $S(G)$, $R(G)$, $Q(G)$, $T(G)$.

and every pair of related white vertices, respectively. Suppose that graphs X and Y have the same vertex set V. Then their *union* is the graph $X \cup Y$ with vertex set V and edge set $E(X) \cup E(Y)$; in particular, we denote by $T(G)$ the union of $R(G)$ and $Q(G)$.

Let G_1 and G_2 be two connected graphs. For convenience, in what follows we denote $V(G_i)$ and $E(G_i)$ by V_i and E_i, $i = 1, 2$, respectively. Next we carry out further operations on these graphs.

Let F be one of the symbols S, R, Q or T. We denote by $G_1 +_F G_2$ the *F-sum* of G_1 and G_2 for which the set of vertices $V(G_1 +_F G_2) = (V_1 \bigcup E_1) \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $u_1 = v_1 \in V_1$ and $u_2v_2 \in E_2$ or $u_2 = v_2$ and $u_1v_1 \in E(F(G_1))$.

Note that $G_1 +_F G_2$ has $|V_2|$ copies of the graph $F(G_1)$, and we may label these copies with vertices of G_2. The vertices in each copy have two situations: the vertices in V_1 which are still referred to as black vertices and the vertices in E_1 which are still referred to as white vertices. Now we join only black vertices with the same name in $F(G_1)$ in which their corresponding labels are adjacent in G_2.

Suppose that x and y are two vertices of a connected graph G. Then the *distance* between x and y, $d(x, y|G)$, is the length of the shortest path between x and y. The following three lemmas are from Ref. [4] and will be used repeatedly in the proof of our main results.

Lemma 2.1. *Let G_1 and G_2 be two connected graphs and $v = (v_1, v_2)$ be a vertex of $G_1 +_F G_2$. Then:*

(a) *If $v_1 \in V_1$ (that is v is a black vertex), then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ we have*

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2).$$

(b) *If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$ with $u_2 \neq v_2, u_1 = u_1^1v_1^1 \in E_1$ and $u_1^1, v_1^1 \in V_1$ (that is v and u are white vertices in different copies of $F(G_1)$), we have*

$$d(u, v|G_1 +_F G_2) = 1 + d(u_2, v_2|G_2) + \min\{d(u_1^1, v_1|F(G_1)), d(v_1^1, v_1|F(G_1))\}.$$

(c) *If $v_1 \in E_1$, then for all $u = (u_1, u_2) \in V(G_1 +_F G_2)$, where $u_2 = v_2$ and $u_1 \in E_1$ (that is v and u are white vertices in the same copy of $F(G_1)$), we have*

$$d(u, v|G_1 +_F G_2) = d(u_1, v_1|F(G_1)).$$

Lemma 2.2. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1, u_2, v_2 \in V_2$ and $F = S$ or R. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$ with $u_2 \neq v_2$, we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

Lemma 2.3. *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E_1, u_2, v_2 \in V_2$ and $F = Q$ or T. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $G_1 +_F G_2$ with $u_2 \neq v_2$, we have*

$$d(u, v|G_1 +_F G_2) = \begin{cases} 2 + d(u_2, v_2|G_2) & \text{if } u_1 = v_1, \\ 1 + d(u_1, v_1|F(G_1)) + d(u_2, v_2|G_2) & \text{if } u_1 \neq v_1. \end{cases}$$

3. The main results

Let $e = uv$ be an edge of a connected graph G. Then we denote by $M_{eu}(e|G)$ (or $M_{ev}(e|G)$) the set of vertices of G lying closer to the vertex u (or v) than to v (or u). If we denote by $|A|$ the cardinality of a set A, and suppose that $m_{eu}(e|G) = |M_{eu}(e|G)|$ and $m_{ev}(e|G) = |M_{ev}(e|G)|$, then the *vertex PI index* of G, PI_v, is defined as the summation of $m_{eu}(e|G) + m_{ev}(e|G)$ over all edges e of G. We denote by $M_e(G)$ the set of vertices of G that are not equidistant from both ends of the edge e and suppose that $m_e(G) = |M_e(G)|$. Then $m_e(G) = m_{eu}(e|G) + m_{ev}(e|G)$ and $PI_v = \sum_{e\in E(G)} m_e(G)$. In this section we will give the explicit expressions for $PI_v(G_1 +_F G_2)$ in terms of other indices of $F(G_1)$ and G_2.

For convenience, we introduce the following notation. Set

$$\mathscr{A} := \{e = uv \in E(G_1 +_F G_2) : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2\}$$
$$\mathscr{B} := \{e = uv \in E(G_1 +_F G_2) : u = (u_1, u_2) \in V_1 \times V_2, v = (v_1, v_2) \in E_1 \times V_2\}$$
$$\mathscr{C} := \{e = uv \in E(G_1 +_F G_2) : u = (u_1, u_2), v = (v_1, v_2) \in E_1 \times V_2\}.$$

Then $E(G_1 +_F G_2) = \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$. Suppose that $\mathcal{A} = \sum_{e\in\mathscr{A}} m_e(G_1 +_F G_2)$, $\mathcal{B} = \sum_{e\in\mathscr{B}} m_e(G_1 +_F G_2)$ and $\mathcal{C} = \sum_{e\in\mathscr{C}} m_e(G_1 +_F G_2)$. Then $PI_v(G_1 +_F G_2) = \mathcal{A} + \mathcal{B} + \mathcal{C}$.

Suppose that e is an edge of a graph G. Then we denote by $\widetilde{M}_e(G)$ the set of vertices of G that are equidistant from both ends of e, and suppose that $\widetilde{m}_e(G) = |\widetilde{M}_e(G)|$ and $\widetilde{m}(G) = \sum_{e\in E(G)} \widetilde{m}_e(G)$.

Theorem 3.1. *Let G_1 and G_2 be two connected graphs. Then*

$$PI_v(G_1 +_S G_2) = (|V_1| + |E_1|)(|V_1|PI_v(G_2) + 2|E_1|\,|V_2|^2).$$

Proof. By the definition of the S-sum, we know that $\mathscr{C} = \emptyset$, and so $\mathcal{C} = 0$. Next we only need to compute $\mathcal{A}$ and $\mathcal{B}$ to obtain $PI_v(G_1 +_S G_2)$.

Suppose that $e = uv \in \mathscr{A}$. Then, by the definition of the S-sum, we know that $u_1 = v_1$ and $e_2 = u_2v_2 \in E(G_2)$. For any $w = (w_1, w_2) \in V(G_1 +_S G_2)$, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_S G_2) = d(w_1, u_1|S(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_S G_2) = d(w_1, v_1|S(G_1)) + d(w_2, v_2|G_2).$$

From the above two equations, we know that $w \in \widetilde{M}_e(G_1 +_S G_2)$ if and only if $w_2 \in \widetilde{M}_{e_2}(G_2)$. Therefore, $m_e(G_1 +_S G_2) = (|V_1| + |E_1|)(|V_2| - \widetilde{m}_{e_2}(G_2))$, and further we obtain

$$\mathcal{A} = |V_1|(|V_1| + |E_1|)PI_v(G_2).$$

Suppose that $e = uv \in \mathscr{B}$. Then, by the definition of the S-sum, we know that $u_2 = v_2$ and u_1 is an end vertex of v_1 in G_1. If $w = (w_1, w_2) \in V_1 \times V_2$ then, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_S G_2) = d(w_1, u_1|S(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_S G_2) = d(w_1, v_1|S(G_1)) + d(w_2, v_2|G_2).$$

Since u_1 is an end vertex of v_1 in G_1, $d(w_1, u_1|S(G_1)) \neq d(w_1, v_1|S(G_1))$. Note that $u_2 = v_2$. From the above two equations, we know that $w \notin \widetilde{M}_e(G_1 +_S G_2)$.

If $w \in E_1 \times V_2$ then, by Lemmas 2.1(a) and 2.2, we have

$$d(w, u|G_1 +_S G_2) = d(w_1, u_1|S(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_S G_2) = \begin{cases} 2 + d(w_2, v_2|G_2) & \text{if } w_1 = v_1, \\ d(w_1, v_1|S(G_1)) + d(w_2, v_2|G_2) & \text{if } w_1 \neq v_1. \end{cases}$$

Note once more that u_1 is an end of v_1 in G_1. We know that $d(w_1, u_1|S(G_1)) = 1$ if $w_1 = v_1$, and $d(w_1, u_1|S(G_1)) \neq d(w_1, v_1|S(G_1))$ otherwise. Since $u_2 = v_2$, from the above two equations, we can see that $w \notin \widetilde{M}_e(G_1 +_S G_2)$.

From the above argument we know that $\widetilde{M}_e(G_1 +_S G_2) = \phi$ if $e \in \mathscr{B}$. Therefore,

$$\begin{aligned}\mathcal{B} &= |V_2|(|V_1| + |E_1|) \cdot 2|E_1|\,|V_2| \\ &= 2|E_1|\,|V_2|^2(|V_1| + |E_1|).\end{aligned}$$

Hence we obtain

$$PI_v(G_1 +_S G_2) = \mathcal{A} + \mathcal{B} = (|V_1| + |E_1|)(|V_1|PI_v(G_2) + 2|E_1|\,|V_2|^2). \quad \square$$

Theorem 3.2. *Let G_1 and G_2 be two connected graphs. Then*

$$PI_v(G_1 +_R G_2) = |V_1|(|V_1| + |E_1|)PI_v(G_2) + |V_2|^2 PI_v(R(G_1)).$$

Proof. By the definition of the R-sum, we know that $\mathscr{C} = \emptyset$, and so $\mathcal{C} = 0$. Next we only need to compute $\mathcal{A}$ and $\mathcal{B}$ to obtain $PI_v(G_1 +_R G_2)$.

Suppose that $e = uv \in \mathscr{A}$. Then we further set

$$\mathscr{A}_1 := \{e = uv : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_1 = v_1 \text{ and } e_2 = u_2v_2 \in E(G_2)\};$$
$$\mathscr{A}_2 := \{e = uv : u = (u_1, u_2), v = (v_1, v_2) \in V_1 \times V_2, u_2 = v_2 \text{ and } e_1 = u_1v_1 \in E(R(G_1))\}.$$

Then $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Suppose that $\mathcal{A}_1 = \sum_{e\in\mathscr{A}_1} m_e(G_1 +_R G_2)$ and $\mathcal{A}_2 = \sum_{e\in\mathscr{A}_2} m_e(G_1 +_R G_2)$. Then $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

As in the former proof of Theorem 3.1, we can see that $\mathcal{A}_1 = |V_1|(|V_1| + |E_1|)PI_v(G_2)$.

For any $e = uv \in \mathscr{A}_2$ and $w = (w_1, w_2) \in V(G_1 +_R G_2)$, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_R G_2) = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_R G_2) = d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2).$$

Since $u_2 = v_2$, we can easily see that $w \in \widetilde{M}_e(G_1 +_R G_2)$ if and only if $w_1 \in \widetilde{M}_{e_1}(R(G_1))$. Therefore, $m_e(G_1 +_R G_2) = |V_2|(|V_1| + |E_1|) - |V_2|\widetilde{m}_{e_1}(R(G_1))$, and we further have

$$\mathcal{A}_2 = |E_1|\,|V_2|^2(|V_1| + |E_1|) - |V_2|^2 \sum_{e_1\in E(R(G_1))} \widetilde{m}_{e_1}(R(G_1)).$$

Hence, we obtain $\mathcal{A} = (|V_1| + |E_1|)(|V_1|PI_v(G_2) + |E_1|\,|V_2|^2) - |V_2|^2 \sum_{e_1\in E(R(G_1))} \widetilde{m}_{e_1}(R(G_1))$.

Suppose that $e = uv \in \mathscr{B}$. Then, by the definition of the R-sum, we know that $u_2 = v_2$, $\widehat{e}_1 = u_1v_1 \in E(R(G_1))$ and u_1 is an end vertex of v_1 in G_1. If $w = (w_1, w_2) \in V_1 \times V_2$ then, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_R G_2) = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_R G_2) = d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2).$$

Since $u_2 = v_2$ and u_1 is an end of v_1 in G_1, from the above equations, we know that $w \in \widetilde{M}_e(G_1 +_R G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(R(G_1))$.

If $w \in E_1 \times V_2$ then, by Lemmas 2.1(a) and 2.2, we have

$$d(w, u|G_1 +_R G_2) = d(w_1, u_1|R(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_R G_2) = \begin{cases} 2 + d(w_2, v_2|G_2) & \text{if } w_1 = v_1, \\ d(w_1, v_1|R(G_1)) + d(w_2, v_2|G_2) & \text{if } w_1 \neq v_1. \end{cases}$$

Since $u_2 = v_2$, from the equations, we observe that if $w_1 = v_1$ then $d(w_1, u_1|S(G_1)) = 1$, and so $w \notin \widetilde{M}_e(G_1 +_R G_2)$; and if $w_1 \neq v_1$ then we have $w \in \widetilde{M}_e(G_1 +_R G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(R(G_1))$. Hence, $m_e(G_1 +_R G_2) = |V_2|(|V_1| + |E_1|) - |V_2|\widetilde{m}_{\widehat{e}_1}(R(G_1))$, which shows that

$$\mathcal{B} = 2|E_1|\,|V_2|^2(|V_1| + |E_1|) - |V_2|^2 \sum_{\widehat{e}_1\in E(R(G_1))} \widetilde{m}_{\widehat{e}_1}(R(G_1)).$$

Therefore, we have

$$PI_v(G_1 +_R G_2) = \mathcal{A} + \mathcal{B} = |V_1|(|V_1| + |E_1|)PI_v(G_2) + |V_2|^2PI_v(R(G_1)). \quad \square$$

Suppose that x and $e = uv$ are a vertex and an edge of a connected graph G, respectively. Then the *distance* from e to x is the smaller of $d(u, x|G)$ and $d(v, x|G)$. Denote by $N_{eu}(e|G)$ (or $N_{ev}(e|G)$) the set of edges in G lying closer to the vertex u (or v) than the vertex v (or u), and suppose that $n_{eu}(e|G) = |N_{eu}(e|G)|$ and $n_{ev}(e|G) = |N_{ev}(e|G)|$. Recall that the *Padmakar–Ivan index* of a graph G, $PI(G)$, is the summation of $n_{eu}(e|G) + n_{ev}(e|G)$ over all the edges $e = uv$ of G. In this definition, edges equidistant from both ends of the edge e are not counted. One of the oldest graph invariants is the first Zagreb index, which was introduced by Gutman and Trinajstić [8], and is defined as $M_1(G) = \sum_{v\in V(G)} deg(v)^2$ for a graph G.

Theorem 3.3. *Let G_1 and G_2 be two connected graphs. Then*

$$PI_v(G_1 +_Q G_2) = |V_1|(|V_1| + |E_1|)PI_v(G_2) - 2|V_2|(|V_2| - 1)PI(G_1) + |V_2|^2PI_v(Q(G_1)) + 2|E_1|^2|V_2|(|V_2| - 1) - |V_2|(|V_2| - 1)M_1(G_1).$$

Proof. Suppose that $e = uv \in \mathscr{A}$. Then, as in the former proof of Theorem 3.1, we can obtain

$$\mathcal{A} = |V_1|(|V_1| + |E_1|)PI_v(G_2).$$

Suppose that $e = uv \in \mathscr{B}$. Then, by the definition of the Q-sum, we know that $u_2 = v_2$, $\widehat{e_1} = u_1v_1 \in E(Q(G_1))$ and u_1 is an end of v_1 in G_1. If $w = (w_1, w_2) \in V_1 \times V_2$ then, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_Q G_2) = d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_Q G_2) = d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).$$

Since $u_2 = v_2$, from the above equations, we observe that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e_1}}(Q(G_1))$. $\square$

Claim 1. *Suppose that $v_1 = u_1u_1'$ is an edge of a connected graph G_1, and suppose that $\widehat{e}_1 = u_1v_1 \in E(Q(G_1))$. Then, for $w_1 \in V_1$, we have $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$ if and only if $w_1 \in M_{v_1u_1}(v_1|G_1) \setminus \{u_1\}$ or $w_1 \in \widetilde{M}_{v_1}(G_1)$.*

Proof. Suppose that $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$. If $w_1 \in M_{v_1u_1'}(v_1|G_1)$ and we let $P_t = x_1x_2\cdots x_t$ be the shortest path between w_1 and u_1' in G_1, then $P_{t+1} = x_1x_2\cdots x_tx_{t+1}$ is the shortest path between w_1 and u_1 in G_1, where $x_1 = w_1, x_t = u_1'$ and $x_{t+1} = u_1$. Thus, by the definition of $Q(G_1)$, we can see that $P'_{t+1} = w_1y_1y_2\cdots y_{t-1}v_1$ is the shortest path between w_1 and v_1 in $Q(G_1)$ and that $P'_{t+2} = w_1y_1y_2\cdots y_{t-1}v_1u_1$ is the shortest path between w_1 and u_1 in $Q(G_1)$, where $y_i = x_ix_{i+1} \in E_1, i = 1, 2, \ldots, t-1$. This contradicts $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$, and so $w_1 \in M_{v_1u_1}(v_1|G_1) \setminus \{u_1\}$ or $w_1 \in \widetilde{M}_{v_1}(G_1)$.

If $w_1 \in M_{v_1u_1}(v_1|G_1) \setminus \{u_1\}$, and we let $P_k = q_1q_2\cdots q_k$ be the shortest path between w_1 and u_1 in G_1, then $P_{k+1} = q_1q_2\cdots q_kq_{k+1}$ is the shortest path between w_1 and u_1' in G_1, where $q_1 = w_1$, $q_k = u_1$ and $q_{k+1} = u_1'$. Thus, by the definition of $Q(G_1)$, we can see that $P'_{k+1} = w_1z_1z_2\cdots z_{k-1}u_1$ is the shortest path between w_1 and u_1 in $Q(G_1)$ and that $P''_{k+1} = w_1z_1z_2\cdots z_{k-1}v_1$ is the shortest path between w_1 and v_1 in $Q(G_1)$, where $z_i = q_iq_{i+1} \in E_1,\ i = 1, 2, \ldots, k-1$. This implies that $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$. Similarly, if $w_1 \in \widetilde{M}_{v_1}(G_1)$, then we can prove that $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$. Hence, Claim 1 is complete.

If $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 = v_2$ then, by Lemma 2.1(a) and (c), we have

$$d(w, u|G_1 +_Q G_2) = d(w_1, u_1|Q(G_1));$$
$$d(w, v|G_1 +_Q G_2) = d(w_1, v_1|Q(G_1)).$$

These two equations show that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$. □

Claim 2. *Suppose that $v_1 = u_1u_1'$ is an edge of a connected graph G_1, and suppose that $\widehat{e}_1 = u_1v_1 \in E(Q(G_1))$. Then, for $w_1 \in E_1$, we have $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$ if and only if $w_1 \in N_{v_1u_1}(v_1|G_1)$ or $w_1 \in \widetilde{N}_{v_1}(G_1) \setminus \{v_1\}$, where $\widetilde{N}_{v_1}(G_1)$ is the set of edges of G_1 that are equidistant from both ends of v_1.*

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 2 is true.

If $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 \neq v_2$ then, by Lemmas 2.1(a) and 2.3, we have

$$d(w, u|G_1 +_Q G_2) = d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_Q G_2) = \begin{cases} 2 + d(w_2, v_2|G_2) & \text{if } w_1 = v_1, \\ 1 + d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2) & \text{if } w_1 \neq v_1. \end{cases}$$

Since u_1 is an end of v_1 in G_1, if $w_1 = v_1$ then $d(w_1, u_1|Q(G_1)) = 1$, and so $w \notin \widetilde{M}_e(G_1 +_Q G_2)$; and if $w_1 \neq v_1$ then, by the definition of $Q(G_1)$, we have $d(w_1, u_1|Q(G_1)) = d(w_1, v_1|Q(G_1))$ or $d(w_1, u_1|Q(G_1)) = 1 + d(w_1, v_1|Q(G_1))$. This implies that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(Q(G_1))$.

Suppose that $\widetilde{n}_{v_1}(G_1) = |\widetilde{N}_{v_1}(G_1)|$. Then, by Claims 1 and 2, we have for $e \in \mathscr{B}$

$$\begin{aligned} m_e(G_1 +_Q G_2) &= |V_2|(|V_1| + |E_1|) - |V_2|(m_{v_1u_1}(v_1|G_1) - 1 + \widetilde{m}_{v_1}(G_1)) - (n_{v_1u_1}(v_1|G_1) + \widetilde{n}_{v_1}(G_1) - 1) \\ &\quad - (|V_2| - 1)(|E_1| - n_{v_1u_1}(v_1|G_1) - \widetilde{n}_{v_1}(G_1)) \\ &= |V_1|\,|V_2| + |V_2| + |E_1| + 1 - |V_2|(m_{v_1u_1}(v_1|G_1) + \widetilde{m}_{v_1}(G_1)) + (|V_2| - 2)(n_{v_1u_1}(v_1|G_1) + \widetilde{n}_{v_1}(G_1)). \end{aligned}$$

Therefore, by the definition of $PI(G_1)$, we obtain

$$\mathscr{B} = 2|E_1|\,|V_2|(|E_1|\,|V_2| + |V_2| - |E_1| + 1) + |V_2|^2 PI_v(G_1) - |V_2|(|V_2| - 2)PI(G_1).$$

Suppose that $e = uv \in \mathscr{C}$. Then by the definition of the Q-sum, we have $u_2 = v_2$, and $\bar{e}_1 = u_1v_1$ is an edge of $Q(G_1)$.

If $w = (w_1, w_2) \in V_1 \times V_2$ then, by Lemma 2.1(a), we have

$$d(w, u|G_1 +_Q G_2) = d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_Q G_2) = d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).$$

We can easily see that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\bar{e}_1}(Q(G_1))$.

If $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 = u_2$ then, by Lemma 2.1(c), we have

$$d(w, u|G_1 +_Q G_2) = d(w_1, u_1|Q(G_1));$$
$$d(w, v|G_1 +_Q G_2) = d(w_1, v_1|Q(G_1)).$$

These two equations show that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\bar{e}_1}(Q(G_1))$.

If $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 \neq u_2$, then we distinguish the following three cases.

Case 1. Suppose that $w_1 = u_1$. Then, by Lemma 2.3, we have

$$d(w, u|G_1 +_Q G_2) = 2 + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_Q G_2) = 1 + d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).$$

Note that $d(w_1, v_1|Q(G_1)) = 1$ since $w_1 = u_1$. So, in this case, $w \in \widetilde{M}_e(G_1 +_Q G_2)$.

Case 2. Suppose that $w_1 = v_1$. Then, as in the proof of Case 1, we can see that $w \in \widetilde{M}_e(G_1 +_Q G_2)$.

Case 3. Suppose that $w_1 \neq v_1$ and $w_1 \neq u_1$. Then, by Lemma 2.3, we have

$$d(w, u|G_1 +_Q G_2) = 1 + d(w_1, u_1|Q(G_1)) + d(w_2, u_2|G_2);$$
$$d(w, v|G_1 +_Q G_2) = 1 + d(w_1, v_1|Q(G_1)) + d(w_2, v_2|G_2).$$

We can easily see that $w \in \widetilde{M}_e(G_1 +_Q G_2)$ if and only if $w_1 \in \widetilde{M}_{\bar{e}_1}(Q(G_1))$.

Thus, by the above argument, we know that for $e \in \mathscr{C}$

$$m_e(G_1 +_Q G_2) = |V_2|(|V_1| + |E_1|) - 2(|V_2| - 1) - |V_2|\widetilde{m}_{\bar{e}_1}(Q(G_1)).$$

Therefore, by the definitions of $M_1(G_1)$ and $PI(G_1)$, combining Claims 1 and 2 we obtain

$$\begin{aligned}\mathcal{C} &= |V_2| \sum_{\bar{e}_1 \in E(Q(G_1))} (|V_2|(|V_1| + |E_1|) - 2(|V_2| - 1)) - |V_2|^2 \sum_{\bar{e}_1 \in E(Q(G_1))} \widetilde{m}_{\bar{e}_1}(Q(G_1)) \\ &= |V_2|^2 \sum_{\bar{e}_1 \in E(Q(G_1))} m_{\bar{e}_1}(Q(G_1)) - 2|V_2|(|V_2| - 1)\left(\frac{1}{2}M_1(G_1\right) - |E_1|) \\ &= |V_2|^2(PI_v(Q(G_1)) - PI(G_1) - PI_v(G_1) - 4|E_1|) - |V_2|(|V_2| - 1)(M_1(G_1) - 2|E_1|).\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}PI_v(G_1 +_Q G_2) = \mathcal{A} + \mathcal{B} + \mathcal{C} &= |V_1|(|V_1| + |E_1|)PI_v(G_2) - 2|V_2|(|V_2| - 1)PI(G_1) + |V_2|^2 PI_v(Q(G_1)) \\ &\quad + 2|E_1|^2|V_2|(|V_2| - 1) - |V_2|(|V_2| - 1)M_1(G_1). \quad \square\end{aligned}$$

Theorem 3.4. *Let G_1 and G_2 be two connected graphs. Then*

$$\begin{aligned}PI_v(G_1 +_T G_2) &= |V_1|(|V_1| + |E_1|)PI_v(G_2) - 2|V_2|(|V_2| - 1)PI(G_1) + |V_2|^2 PI_v(T(G_1)) - |V_2|(|V_2| - 1)M_1(G_1) \\ &\quad + 2|E_1|^2|V_2|(|V_2| - 1).\end{aligned}$$

Proof. Suppose that $e = uv \in \mathscr{A}$, and suppose that $e_1 = u_1v_1$. Then, as in the former proof of Theorem 3.2, we have

$$\begin{aligned}\mathcal{A} &= (|V_1| + |E_1|)(|V_1|PI_v(G_2) + |E_1|\,|V_2|^2) - |V_2|^2 \sum_{e_1 \in E(T(G_1))} \widetilde{m}_{e_1}(T(G_1)) \\ &= |V_1|(|V_1| + |E_1|)PI_v(G_2) + |V_2|^2 \sum_{e_1 \in E(T(G_1))} m_{e_1}(T(G_1)).\end{aligned}$$

Suppose that $e = uv \in \mathscr{B}$. If $w = (w_1, w_2) \in V_1 \times V_2$, then, like in the middle part of the proof of Theorem 3.3, we can observe that $w \in \widetilde{M}_e(G_1 +_T G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(T(G_1))$. □

Claim 3. *Suppose that $v_1 = u_1u_1'$ is an edge of a connected graph G_1, and suppose that $\widehat{e}_1 = u_1v_1 \in E(T(G_1))$. Then, for $w_1 \in V_1$, we have $w_1 \in \widetilde{M}_{\widehat{e}_1}(T(G_1))$ if and only if $w_1 \in M_{v_1u_1'}(v_1|G_1)$.*

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 3 is true.

If $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 = v_2$, then like in the middle part of the proof of Theorem 3.3, we can see that $w \in \widetilde{M}_e(G_1 +_T G_2)$ if and only if $w_1 \in \widetilde{M}_{\widehat{e}_1}(T(G_1))$. □

Claim 4. *Suppose that $v_1 = u_1u_1'$ is an edge of a connected graph G_1, and suppose that $\widehat{e}_1 = u_1v_1 \in E(T(G_1))$. Then, for $w_1 \in E_1$, we have $w_1 \in \widetilde{M}_{\widehat{e}_1}(T(G_1))$ if and only if $w_1 \in N_{v_1u_1}(v_1|G_1)$ or $w_1 \in \widetilde{N}_{v_1}(G_1) \setminus \{v_1\}$.*

Proof. By a method similar to that of the proof of Claim 1, we can prove that Claim 4 is true.

Suppose that $w = (w_1, w_2) \in E_1 \times V_2$ and $w_2 \neq u_2$. Like in the middle part of the proof of Theorem 3.3, we can obtain $w \notin \widetilde{M}_e(G_1 +_T G_2)$ if $w_1 = v_1$, and otherwise $w \in \widetilde{M}_e(G_1 +_T G_2)$ if and only if $w_1 \in \widetilde{M}_{\bar{e}_1}(T(G_1))$.

Thus, by Claims 3 and 4, we have for $e \in \mathscr{B}$

$$\begin{aligned}m_e(G_1 +_T G_2) &= |V_2|(|V_1| + |E_1|) - |V_2|m_{v_1u_1'}(v_1|G_1) + 1 + n_{v_1u_1'}(v_1|G_1) - (|V_2| - 1)n_{v_1u_1'}(v_1|G) - |E_1| \\ &= |V_2|(|V_1| + |E_1|) - |E_1| + 1 - |V_2|m_{v_1u_1'}(v_1|G_1) - (|V_2| - 2)n_{v_1u_1'}(v_1|G_1).\end{aligned}$$

Therefore, by the definition of $PI(G_1)$, we obtain

$$\mathcal{B} = 2|E_1|\,|V_2|(|V_1|\,|V_2| + |E_1|\,|V_2| - |E_1| + 1) - |V_2|^2 PI_v(G_1) - |V_2|(|V_2| - 2)PI(G_1).$$

Suppose that $e = uv \in \mathscr{C}$, and suppose that $\bar{e}_1 = u_1v_1$. Then, as in the last part of the proof of Theorem 3.3, we have

$$\mathcal{C} = |V_2|^2 \sum_{\bar{e}_1 \in E(T(G_1))} m_{\bar{e}_1}(T(G_1)) - |V_2|(|V_2| - 1)(M_1(G_1) - 2|E_1|).$$

Hence, we finally obtain

$$\begin{aligned} PI_v(G_1 +_T G_2) &= \mathcal{A} + \mathcal{B} + \mathcal{C} \\ &= |V_1|(|V_1| + |E_1|)PI_v(G_2) - 2|V_2|(|V_2| - 1)PI(G_1) + |V_2|^2 PI_v(T(G_1)) - |V_2|(|V_2| - 1)M_1(G_1) \\ &\quad + 2|E_1|^2|V_2|(|V_2| - 1). \quad \square \end{aligned}$$

4. Postscript

The draft manuscript of a paper was published in Ars Combin. 98 (2011) without the present authors being aware of this. Now we give a simple example to show that the main results in [15] are wrong. Let G_1 and G_2 be the paths on three and two vertices, respectively. Then, by the definition of the vertex PI index, we can easily see that $PI_v(G_1 +_Q G_2) = 106$ and $PI_v(G_1 +_T G_2) = 122$. But using the corresponding formulae in [15], we have $PI_v(G_1 +_Q G_2) = 110$ and $PI_v(G_1 +_T G_2) = 222$.

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